9-4 Arithmetic Series

Content Standard

Extends F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

Objective To define arithmetic series and find their sums

Just as you found formulas for terms of sequences, you can find formulas for the sums of the terms of sequences.

Essential Understanding When you know two terms and the number of terms in a finite arithmetic sequence, you can find the sum of the terms.

A **series** is the indicated sum of the terms of a sequence. A **finite series**, like a finite sequence, has a first term and a last term, while an **infinite series** continues without end.

Finite sequence	**Finite series**
6, 9, 12, 15, 18 ends	$6 + 9 + 12 + 15 + 18$ (The sum is 60.)
Infinite sequence	**Infinite series**
3, 7, 11, 15, . . .	$3 + 7 + 11 + 15 + \dots$

An **arithmetic series** is a series whose terms form an arithmetic sequence (as shown above). When a series has a finite number of terms, you can use a formula involving the first and last term to evaluate the sum.

Take note

Property Sum of a Finite Arithmetic Series

The sum S_n of a finite arithmetic series $a_1 + a_2 + a_3 + \cdots + a_n$ is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term, a_n is the nth term, and n is the number of terms.

Problem 1 Finding the Sum of a Finite Arithmetic Series

What is the sum of the even integers from 2 to 100?

The series $2 + 4 + 6 + \cdots + 100$ is arithmetic with first term 2, last (and 50th) term 100, and common difference 2. The sum is

$$S_{50} = \frac{50}{2}(2 + 100) = 25(102) = 2550.$$

Got It? 1. a. What is the sum of the finite arithmetic series $4 + 9 + 14 + 19 + 24 + \cdots + 99$?

$\frac{n}{2}(a_1 + a_n) = \frac{20}{2}(4+99)$

$10(103)$

1030

$a_n = a_1 + (n-1)d$

$99 = 4 + (n-1)5$

$99 = 4 + 5n - 5$

$99 = 5n - 1$

$100 = 5n$

$20 = n$

Problem 2 Using the Sum of a Finite Arithmetic Series

Bonus A company pays a \$10,000 bonus to salespeople at the end of their first 50 weeks if they make 10 sales in their first week, and then improve their sales numbers by two each week thereafter. One salesperson qualified for the bonus with the minimum possible number of sales. How many sales did the salesperson make in week 50? In all 50 weeks?

10, 12, 14

Think	Write
The first week sales were 10. Sales increased by 2 each week.	$a_1 = 10$ $d = 2$
The sequence is arithmetic. Use the explicit formula to find the sales in week 50. Substitute 50 for n, 10 for a_1, and 2 for d. Then simplify.	$a_n = a_1 + (n-1)d$ $a_{50} = 10 + (50-1)2$ $= 10 + (49)2$ $= 10 + 98 = 108$
Use the formula for S_n to find the total sales for all 50 weeks. Substitute 50 for n, 10 for a_1, and 108 for a_{50}. Then simplify.	$S_n = \frac{n}{2}(a_1 + a_n)$ $S_{50} = \frac{50}{2}(10 + 108)$ $= 25(118) = 2950$
Write the answers.	The salesperson made 108 sales in week 50 and 2950 sales in all 50 weeks.

1 — 50

Got It? 2. The company in Problem 2 has an alternative bonus plan. It pays a \$5000 bonus if a new salesperson makes 10 sales in the first week and then improves by *one* sale per week each week thereafter. One salesperson qualified for this bonus with the minimum number of sales. How many sales did the salesperson make in week 50? In all 50 weeks?

$10, 11, 12 \rightarrow 50$

$a_{50} = 10 + (n-1)(1)$

$10 + (50-1)(1)$

$a_{50} = 10 + 49$

$a_{50} = 59$

$\frac{n}{2}(a_1 + a_n)$

$\frac{50}{2}(10 + 59)$

$25(69)$

1725

Problem 3 Writing a Series in Summation Notation

Multiple Choice What is summation notation for the series?

$7 + 11 + 15 + \cdots + 203 + 207$

Ⓐ $\sum_{n=1}^{51}(4n+3)$ Ⓑ $\sum_{n=1}^{50}(4n+3)$ Ⓒ $\sum_{n=1}^{50}(7n)$ Ⓓ $\sum_{n=1}^{51}(7n)$

The sequence 7, 11, 15, . . . , 203, 207 is arithmetic with first term $a = 7$ and common difference $d = 4$.

$a_n = a_1 + (n-1)d$ Use the explicit formula for an arithmetic sequence.

$a_n = 7 + (n-1)4$ Substitute 7 for a_1, and 4 for d.

$= 4n + 3$ Simplify.

An explicit formula for the *n*th term is $4n + 3$.

$a_n = 4n + 3$ Use the explicit formula to find the value of n for the term 207.

$207 = 4n + 3$ Substitute 207 for a_n.

$204 = 4n$ Solve for n.

$51 = n$

The upper limit is 51. You can write the series as $\sum_{n=1}^{51}(4n+3)$. The correct answer is A.

$a_n = 500 + (n-1)(-10)$
$500 - 10n + 10$
$-10n + 510$

Got It? 3. What is summation notation for the series?

a. $-5 + 2 + 9 + 16 + \cdots + 261 + 268$

b. $500 + 490 + 480 + \cdots + 20 + 10$

b) $\sum_{n=1}^{50} -10n + 510$

$\sum_{n=1}^{40} 7n - 12$

$a_n = a_1 + (n-1)d$
$-5 + (n-1)7$
$-5 + 7n - 7$
$7n - 12$
$268 = 7n - 12$
$280 = 7n$ $n = 40$

$10 = -10n + 510$
$-500 = -10n$
$50 = n$

Take note

Key Concept Summation Notation and Linear Functions

If the explicit formula for the *n*th term in summation notation is a *linear* function of n, then the series is arithmetic. The slope of the linear function is the common difference between terms of the series.

Problem 4 Finding the Sum of a Series

What is the sum of the series written in summation notation?

A $\sum_{n=1}^{70}(5n + 3)$

$a_1 + a_2 + a_3 + a_4 \ldots a_{70}$
8 353

$\frac{70}{2}(8 + 353)$
12635

a. $\sum_{n=1}^{40}(3n - 8)$

$\frac{n}{2}(a_1 + a_n)$

$\frac{40}{2}(-5 + 112)$

2140

Name________________________________

Practice and Problem-Solving Exercises

Find the sum of each finite arithmetic series. See Problem 1.

8. $2 + 4 + 6 + 8$

9. $8 + 9 + 10 + \cdots + 15$

14. Grades A student has taken three math tests so far this semester. His scores for the first three tests were 75, 79, and 83. See Problem 2.

a. Suppose his test scores continue to improve at the same rate. What will be his grade on the sixth (and final) test?

b. What will be his total score for all six tests?

Write each arithmetic series in summation notation. See Problem 3.

15. $4 + 8 + 12 + 16 + 20$

16. $7 + 9 + 11 + \cdots + 21$

Find the sum of each finite series. See Problem 4.

21. $\sum_{n=1}^{5}(2n - 1)$

22. $\sum_{n=1}^{10}(3n - 4)$

23. $\sum_{n=1}^{8}(7 - n)$

Use a graphing calculator to find the sum of each series. See Problem 5.

27. $\sum_{n=1}^{50}(2n - 3)$

28. $\cancel{\sum_{n=1}^{26}(n^2 - 3n)}$

29. $\cancel{\sum_{n=1}^{10}(-2)^n}$