13-2 Angles and the Unit Circle

Content Standard

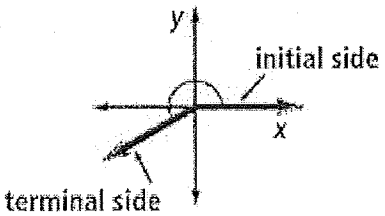
Prepares for F.TF.2 Explain how the unit circle . . . enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Objectives To work with angles in standard position
To find coordinates of points on the unit circle

An angle in the coordinate plane is in **standard position** when the vertex is at the origin and one ray is on the positive x-axis. The ray on the x-axis is the **initial side** of the angle. The other ray is the **terminal side** of the angle.

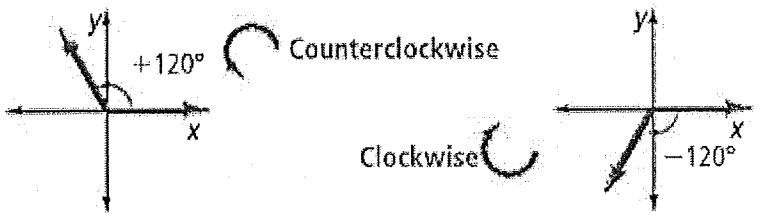
The measure of an angle in standard position is the amount of rotation from the initial side to the terminal side.

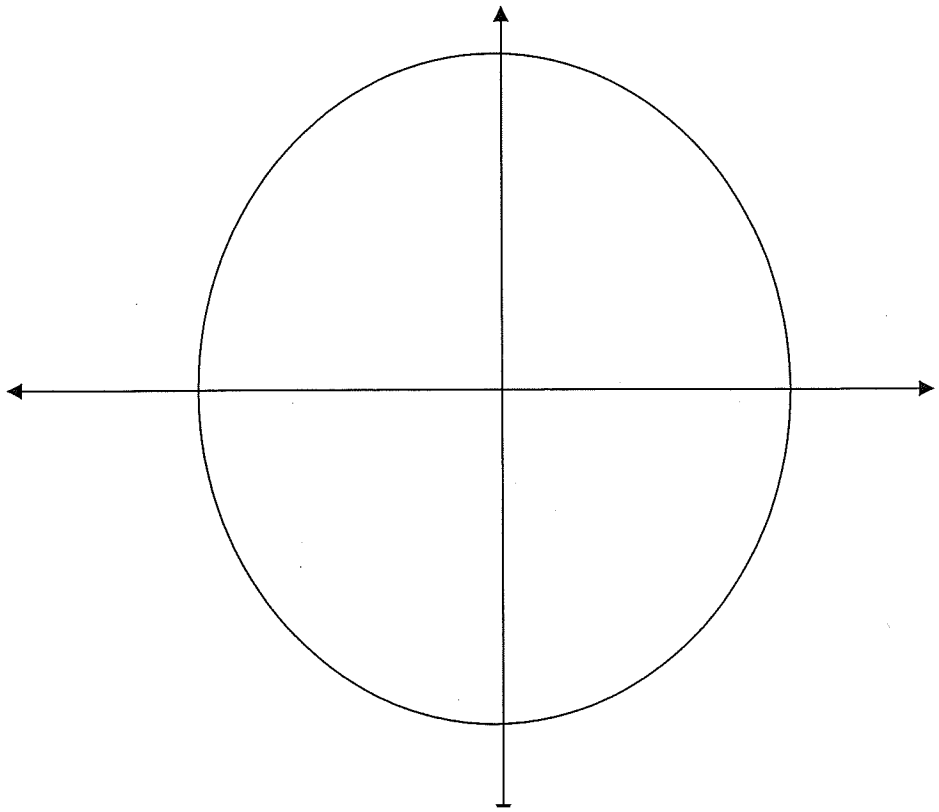
Standard Position



Essential Understanding The measure of an angle in standard position is the input for two important functions. The outputs are the coordinates (called *cosine* and *sine*) of the point on the terminal side of the angle that is 1 unit from the origin.

The measure of an angle is positive when the rotation from the initial side to the terminal side is in the counterclockwise direction. The measure is negative when the rotation is clockwise.





Problem 2 Sketching Angles in Standard Position

What is a sketch of each angle in standard position?

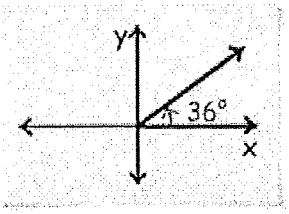
A 36°

B 315°

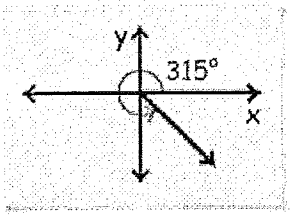
C −150°

Think

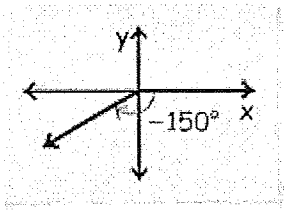
What is the initial side of the angle?
In standard position, the initial side is always the positive *x*-axis.



36° Counterclockwise



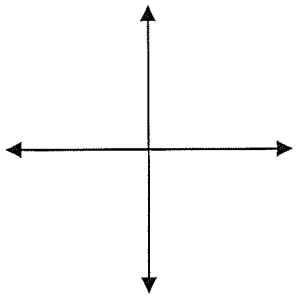
315° Counterclockwise

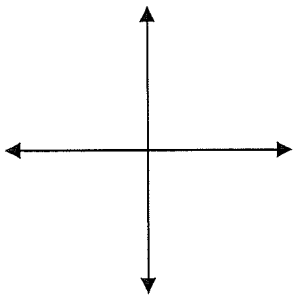


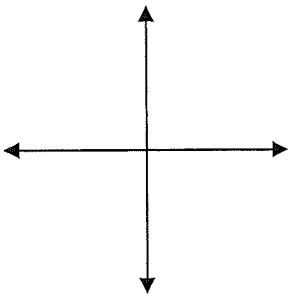
150° Clockwise

Got It? 2. What is a sketch of each angle in standard position?

a. 85° b. −320° c. 180°







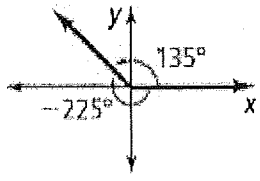
Now try these……

Sketch each angle in standard position. See Problem 2.

13. 40° 14. −130° 15. −270° 16. 120° 17. 95°

Two angles in standard position are **coterminal angles** if they have the same terminal side.

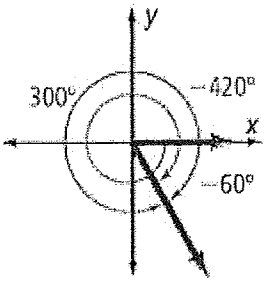
Angles in standard position that have measures 135° and −225° are coterminal.

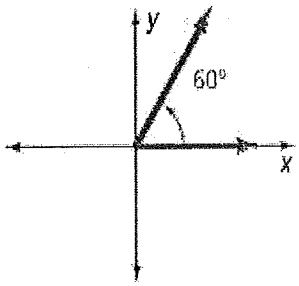


Problem 3 Identifying Coterminal Angles

Multiple Choice Which of the following angles is not coterminal with any of the other three?

(A) 300° (B) −60° (C) 60° (D) −420°





Think

How do you know if two angles are coterminal?
Two angles are coterminal if they differ by a multiple of 360°.

Angles of 300° and −60° are coterminal. An angle of −420° is coterminal with both, since it is a full 360° rotation beyond −60°.

An angle of 60° is not coterminal with any of the other three.

Angles of 300°, −60°, and −420° all have the same terminal side and are coterminal. The 60° angle has a different terminal side. The correct answer is C.

Got It? 3. Which angles are coterminal?

a. −315° b. 45° c. 315° d. 405°

Find the measure of an angle between 0° and 360° coterminal with each given angle.

See Problem 3.

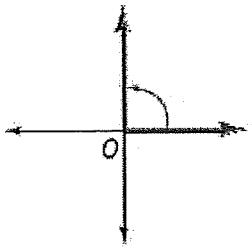
18. 385° 19. 575° 20. −405° 21. −356°

22. 500° 23. −210° 24. 415° 25. −180°

Problem 1 Measuring Angles in Standard Position

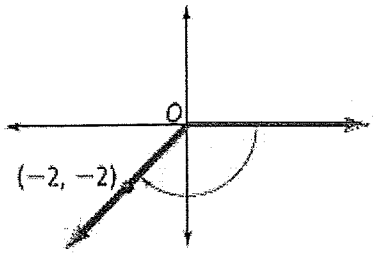
What are the measures of each angle?

A



This angle is a counterclockwise rotation that makes a right angle, so its measure is 90°.

B

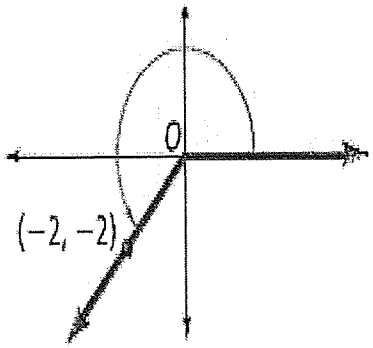


This angle is a clockwise rotation that goes 45° beyond a right angle, so its measure is −135°.

Think

How many degrees are in a circle?
There are 360° in a circle, 180° in half of a circle, and 90° in a quarter of a circle.

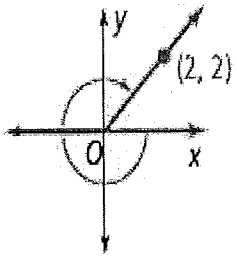
Got It? 1. What is the measure of the angle shown?



Now try these.....

Find the measure of each angle in standard position. See Problem 1.

7.



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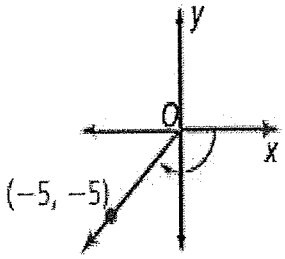

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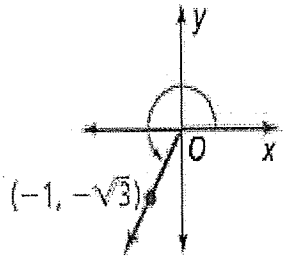

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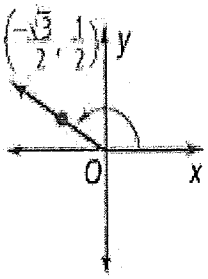

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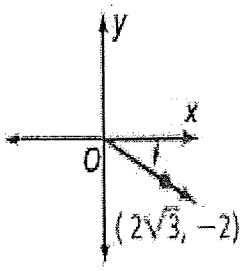

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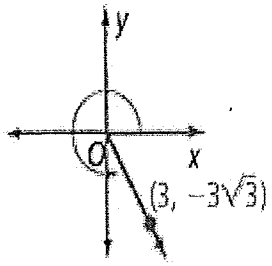
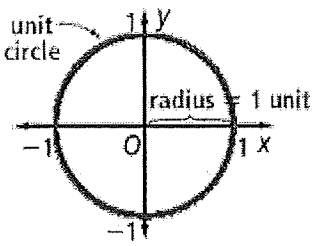


In a 360° angle, a point 1 unit from the origin on the terminal ray makes one full rotation about the origin. The resulting circle is a unit circle. The **unit circle** has a radius of 1 unit and its center at the origin of the coordinate plane. Any right triangle formed by the radius of the unit circle has a hypotenuse of 1. Points on the unit circle are related to periodic functions.

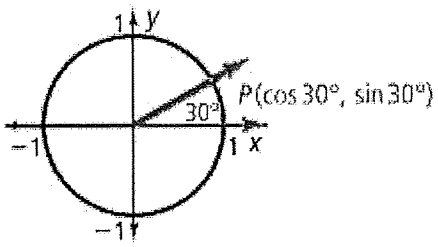


You can use the symbol θ for the measure of an angle in standard position.

Take Note

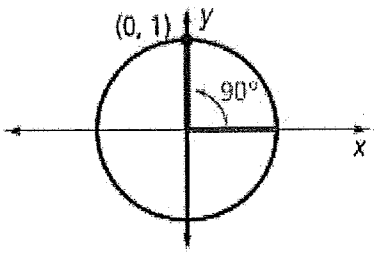
Key Concepts Cosine and Sine of an Angle

Suppose an angle in standard position has measure θ. The **cosine of θ** ($\cos\theta$) is the x-coordinate of the point at which the terminal side of the angle intersects the unit circle. The **sine of θ** ($\sin\theta$) is the y-coordinate.

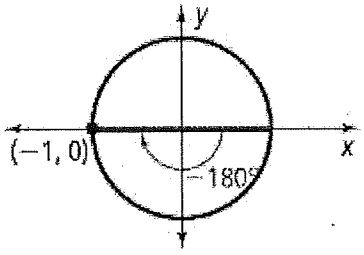


Problem 4 Finding Cosines and Sines of Angles

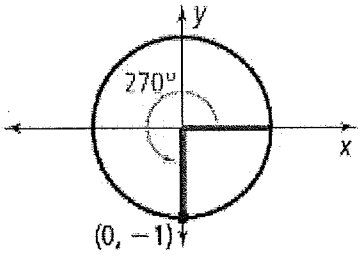
What are $\cos\theta$ and $\sin\theta$ for $\theta = 90°$, $\theta = -180°$, and $\theta = 270°$?



$\cos 90° = 0$
$\sin 90° = 1$



$\cos(-180°) = -1$
$\sin(-180°) = 0$



$\cos 270° = 0$
$\sin 270° = -1$

Got It? **4. a.** What are $\cos\theta$ and $\sin\theta$ for $\theta = -90°$, $\theta = 360°$, and $\theta = 540°$?

b. In a triangle, sine and cosine are ratios between side lengths. What ratios produce the values in (a)?

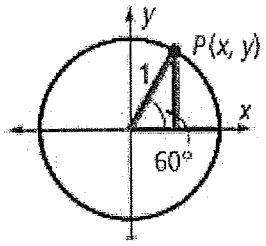
Problem 5 Finding Exact Values of Cosine and Sine

What are the cosine and sine of the angle?

A $\theta = 60°$

Know	Need	Plan
An angle	The x- and y-coordinates of the point where the angle intersects the unit circle	• Sketch the angle on the unit circle. • Use the angle to draw a right triangle with one leg on the x-axis.

The cosine of 60° is the length of the shorter leg of the triangle. The sine of 60° is the length of the longer leg of the triangle. In a 30°-60°-90° triangle, the shorter leg is half the hypotenuse and the longer leg is $\sqrt{3}$ times the shorter leg.



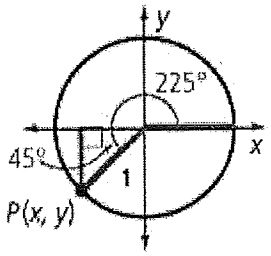
$\cos 60° = x =$ length of shorter leg $= \frac{1}{2}$

$\sin 60° = y =$ length of longer leg $= \frac{\sqrt{3}}{2}$

$\cos 225° = x = -$length of leg $= -\frac{\sqrt{2}}{2}$ $\sin 225° = y = -$length of leg $= -\frac{\sqrt{2}}{2}$

B $\theta = 225°$

Draw the angle in standard position to determine the point $P(x, y)$ on the unit circle. P is in the third quadrant, so the signs of x and y will be negative. Form a right triangle with hypotenuse 1. In a 45°-45°-90° triangle, the lengths of the legs of the triangle are $\frac{\sqrt{2}}{2}$ times the hypotenuse.



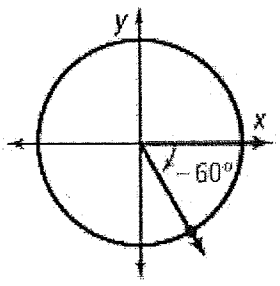
Got It? 5. What are the cosine and sine of the angle?

a. $\theta = -45°$ b. $\theta = 150°$

Find the exact values of the cosine and sine of each angle. Then find the decimal values. Round your answers to the nearest hundredth.

See Problems 4 and 5.

26.



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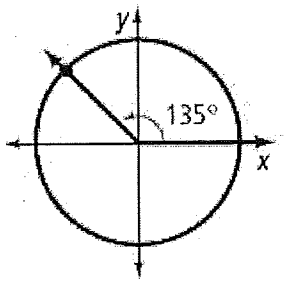

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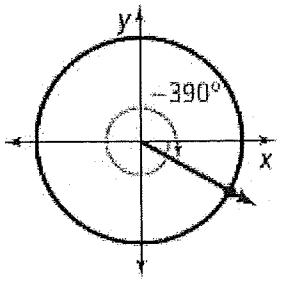

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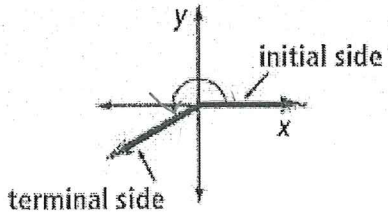
Standard Position



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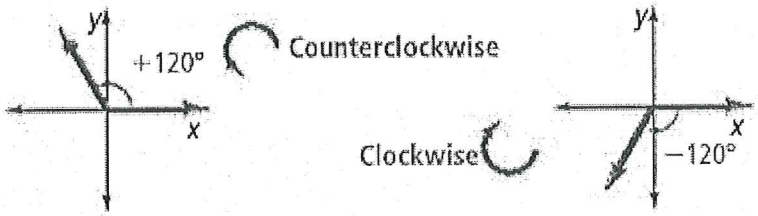

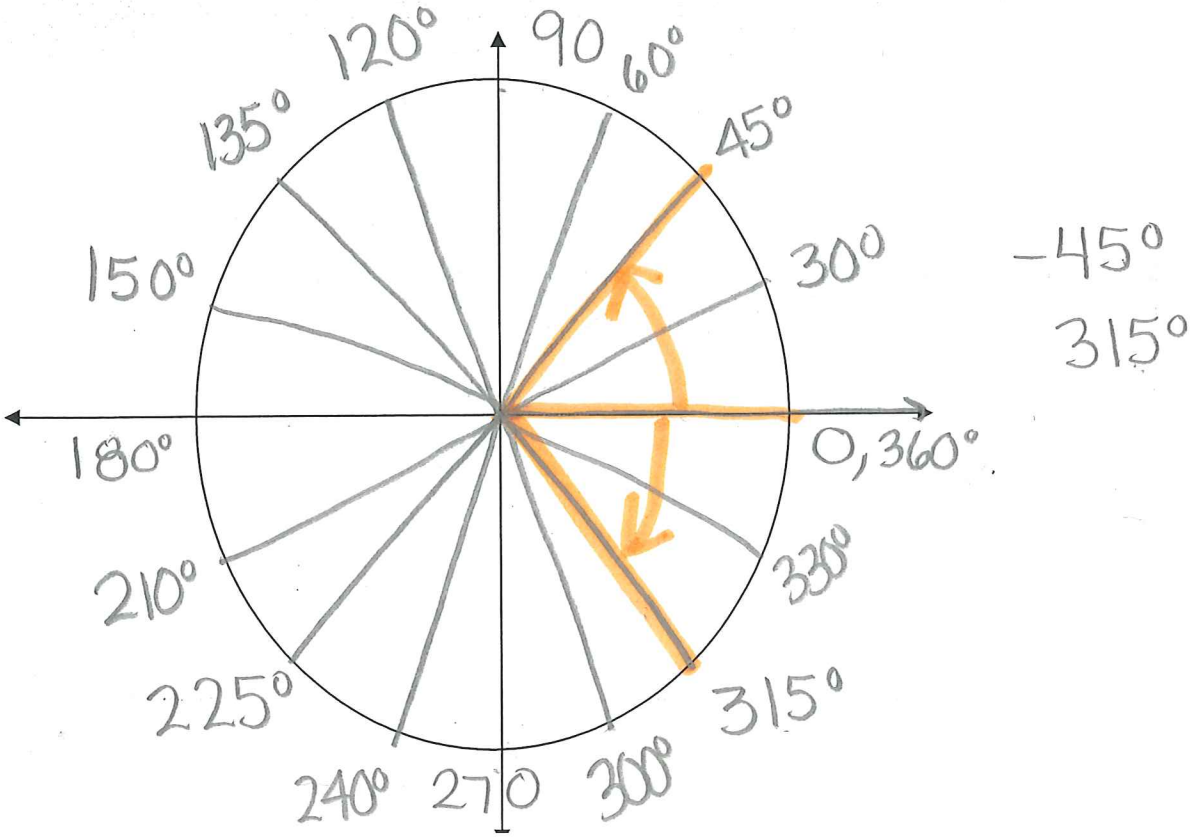

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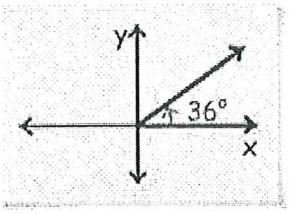
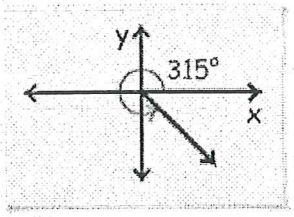


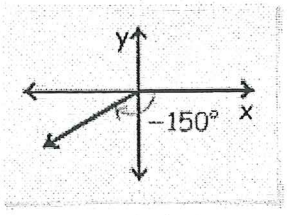
36° Counterclockwise

B 315°



315° Counterclockwise

C −150°



150° Clockwise

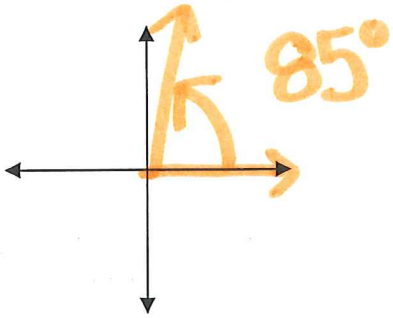
Think

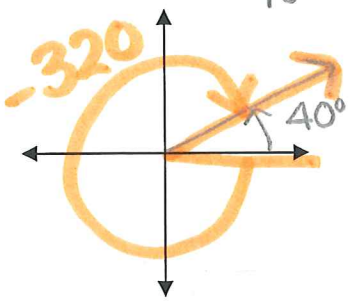
What is the initial side of the angle?
In standard position, the initial side is always the positive *x*-axis.

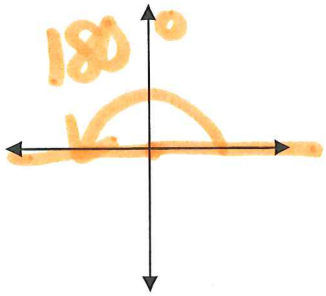
Got It? 2. What is a sketch of each angle in standard position?

a. 85° b. −320° c. 180°

+360
40



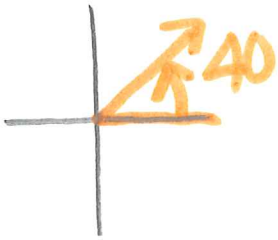




Now try these……

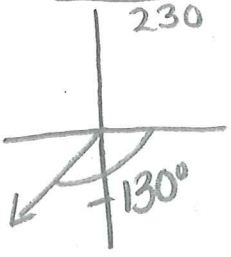
Sketch each angle in standard position. See Problem 2.

13. 40°



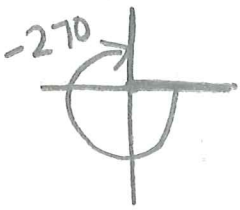
14. −130°

360
230

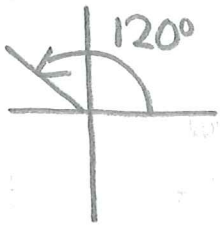


15. −270°

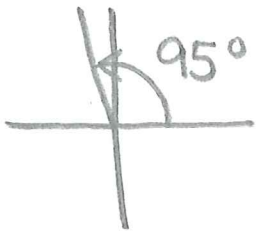
360
90



16. 120°

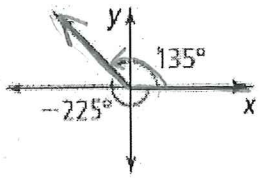


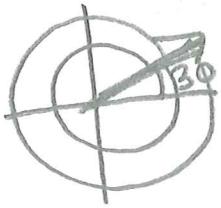
17. 95°



Two angles in standard position are **coterminal angles** if they have the same terminal side.

Angles in standard position that have measures 135° and −225° are coterminal.





3900
360
750°

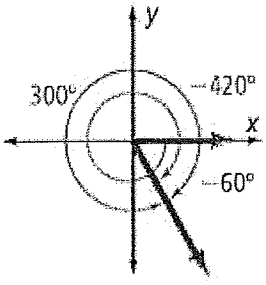
135°
−360
−225°

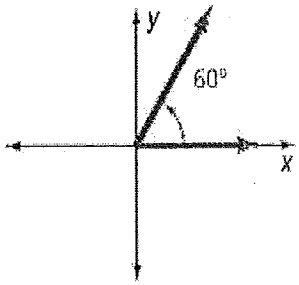
135° is coterminal −225°

Problem 3 Identifying Coterminal Angles

Multiple Choice Which of the following angles is not coterminal with any of the other three?

(A) 300° (B) −60° (C) 60° (D) −420°





Think

How do you know if two angles are coterminal?
Two angles are coterminal if they differ by a multiple of 360°.

Angles of 300° and −60° are coterminal. An angle of −420° is coterminal with both, since it is a full 360° rotation beyond −60°.

An angle of 60° is not coterminal with any of the other three.

Angles of 300°, −60°, and −420° all have the same terminal side and are coterminal. The 60° angle has a different terminal side. The correct answer is C.

Got It? 3. Which angles are coterminal?

a. −315° b. 45° c. 315° d. 405°

(0, 360)

Find the measure of an angle between 0° and 360° coterminal with each given angle.

See Problem 3.

18. 385°
−360
25°

19. 575°
−360
215°

20. −405°
+360
−45
+360
315°

21. −356°
+360
4°

22. 500°
−360
140°

23. −210°
+360
150°

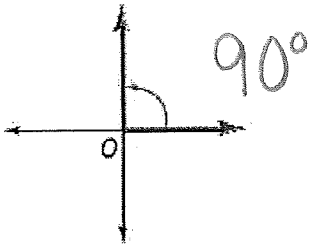
24. 415°
−360°
55°

25. −180°
+360
180°

Problem 1 Measuring Angles in Standard Position

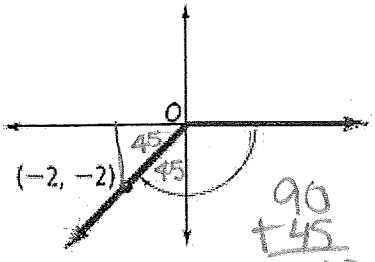
What are the measures of each angle?

A



This angle is a counterclockwise rotation that makes a right angle, so its measure is 90°.

B



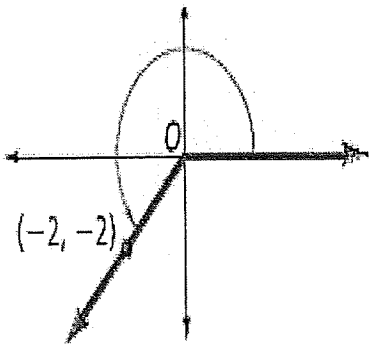
90 + 45 = 135

This angle is a clockwise rotation that goes 45° beyond a right angle, so its measure is −135°.

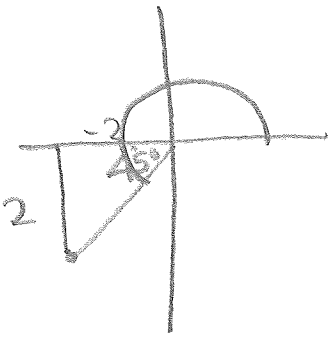
Think

How many degrees are in a circle?
There are 360° in a circle, 180° in half of a circle, and 90° in a quarter of a circle.

Got It? 1. What is the measure of the angle shown?



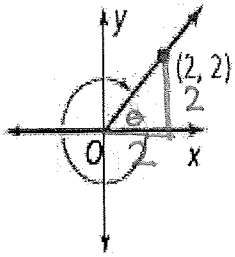
180 + 45 = 225°



Now try these.....

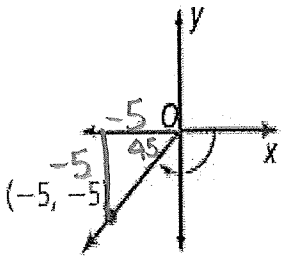
Find the measure of each angle in standard position. See Problem 1.

7.



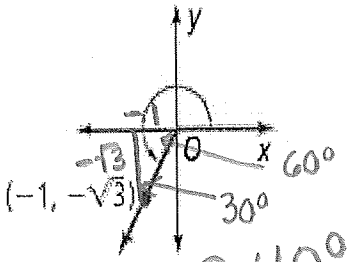
45 − 360 = −315°

8.



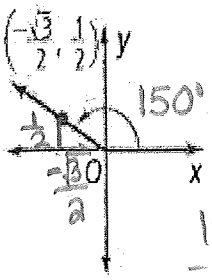
−135

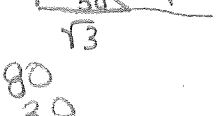
9.



240°

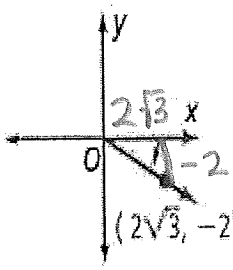
10.

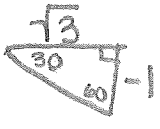




180 − 30 = 150°

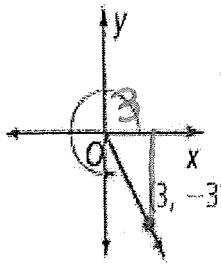
11.





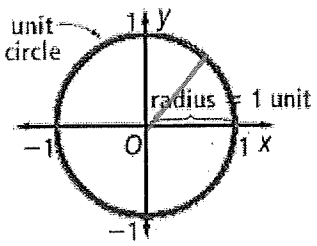
−300°

12.



300°

In a 360° angle, a point 1 unit from the origin on the terminal ray makes one full rotation about the origin. The resulting circle is a unit circle. The **unit circle** has a radius of 1 unit and its center at the origin of the coordinate plane. Any right triangle formed by the radius of the unit circle has a hypotenuse of 1. Points on the unit circle are related to periodic functions.

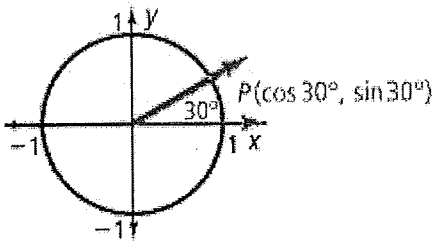


You can use the symbol θ for the measure of an angle in standard position.

Take Note

Key Concepts Cosine and Sine of an Angle

Suppose an angle in standard position has measure θ. The **cosine of θ** ($\cos\theta$) is the x-coordinate of the point at which the terminal side of the angle intersects the unit circle. The **sine of θ** ($\sin\theta$) is the y-coordinate.

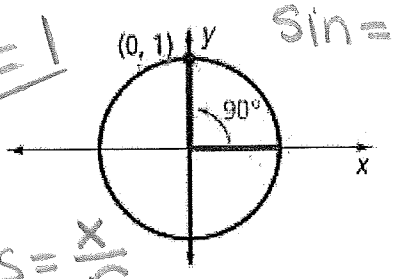


Problem 4 Finding Cosines and Sines of Angles

What are $\cos\theta$ and $\sin\theta$ for $\theta = 90°$, $\theta = -180°$, and $\theta = 270°$?

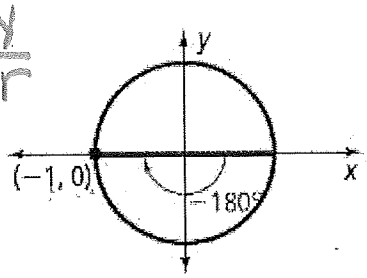
r = 1

Sin = y/r



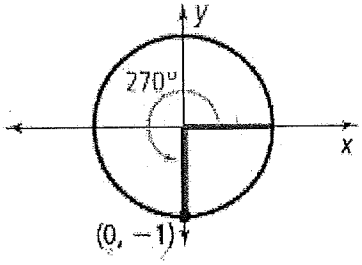
Cos = x/r

$\cos 90° = 0$
$\sin 90° = 1$



$\cos(-180°) = -1$
$\sin(-180°) = 0$

$\sin = \frac{y}{r} = \frac{0}{1} = 0$



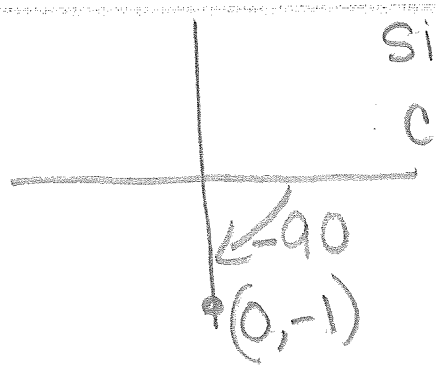
$\cos 270° = 0$
$\sin 270° = -1$

Got It? 4. **a.** What are $\cos\theta$ and $\sin\theta$ for $\theta = -90°$, $\theta = 360°$, and $\theta = 540°$?

b. In a triangle, sine and cosine are ratios between side lengths. What ratios produce the values in (a)?

$\sin -90 = \frac{y}{r} = \frac{-1}{1} = -1$

$\cos -90 = \frac{x}{r} = \frac{0}{1} = 0$



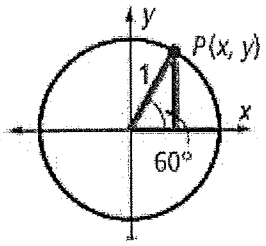
Problem 5 Finding Exact Values of Cosine and Sine

What are the cosine and sine of the angle?

A $\theta = 60°$

Know	Need	Plan
An angle	The x- and y-coordinates of the point where the angle intersects the unit circle	• Sketch the angle on the unit circle. • Use the angle to draw a right triangle with one leg on the x-axis.

The cosine of 60° is the length of the shorter leg of the triangle. The sine of 60° is the length of the longer leg of the triangle. In a 30°-60°-90° triangle, the shorter leg is half the hypotenuse and the longer leg is $\sqrt{3}$ times the shorter leg.



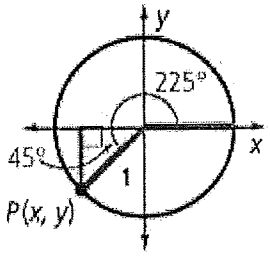
$\cos 60° = x =$ length of shorter leg $= \frac{1}{2}$

$\sin 60° = y =$ length of longer leg $= \frac{\sqrt{3}}{2}$

$\cos 225° = x = -$length of leg $= -\frac{\sqrt{2}}{2}$ $\sin 225° = y = -$length of leg $= -\frac{\sqrt{2}}{2}$

B $\theta = 225°$

Draw the angle in standard position to determine the point $P(x, y)$ on the unit circle. P is in the third quadrant, so the signs of x and y will be negative. Form a right triangle with hypotenuse 1. In a 45°-45°-90° triangle, the lengths of the legs of the triangle are $\frac{\sqrt{2}}{2}$ times the hypotenuse.



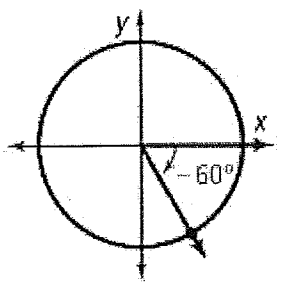
Got It? 5. What are the cosine and sine of the angle?

a. $\theta = -45°$ b. $\theta = 150°$

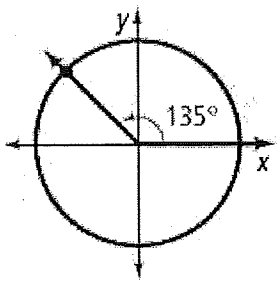
Find the exact values of the cosine and sine of each angle. Then find the decimal values. Round your answers to the nearest hundredth.

See Problems 4 and 5.

26.



27.



28.