3rd Hour

6.4 Trigonometric Functions

Objectives

- Define the trigonometric ratios in the coordinate plane
- Define the trigonometric functions in terms of the unit circle

Extending the Trigonometric Ratios

Trigonometric ratios were defined for acute angles in Section 6.1. The next step is to develop a definition of these ratios that applies to angles of any measure.

To do this, first consider an acute angle θ in standard position. Choose a point P, with coordinates (x, y), on the terminal side, and draw a right triangle, as shown in Figure 6.4-1. The side adjacent to θ has length x and the side opposite θ has length y. The length of the hypotenuse, r, is the distance from the origin, which may be found by using the Pythagorean Theorem.

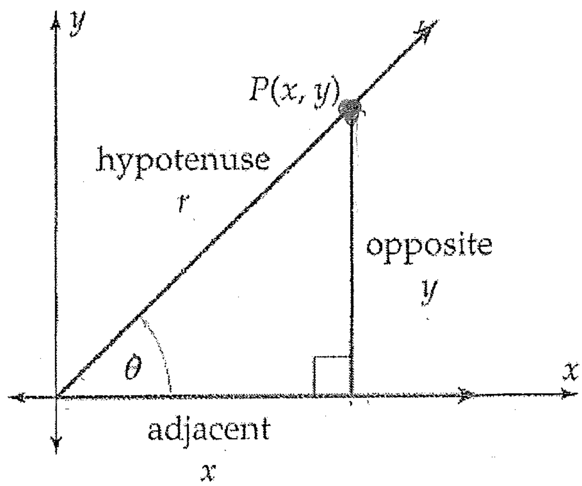


Figure 6.4-1

$$x^2 + y^2 = r^2$$
$$r = \sqrt{x^2 + y^2}$$

$(-3, 1)$

$\sqrt{-3^2 + 1^2} = \sqrt{10}$

The trigonometric ratios can now be written in terms of x, y, and r. For example,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}.$$

NOTE P can be any point on the terminal side of the angle, except for the origin, since different choices for P generate similar right triangles. Thus, the value of a trigonometric ratio depends only on the angle.

Thus, the trigonometric ratios can be described without triangles by using a point on the terminal side of the angle. More importantly, this process can be carried out for *any* angle, not just acute angles. Therefore, the following definition applies to any angle and agrees with the previous definition when the angle is acute.

Trigonometric Ratios in the Coordinate Plane

Let θ be an angle in standard position and let $P(x, y)$ be any point on the terminal side of θ. Let r be the distance from (x, y) to the origin:

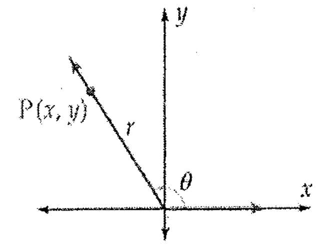
$$r = \sqrt{x^2 + y^2}$$

Then the trigonometric ratios of θ are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

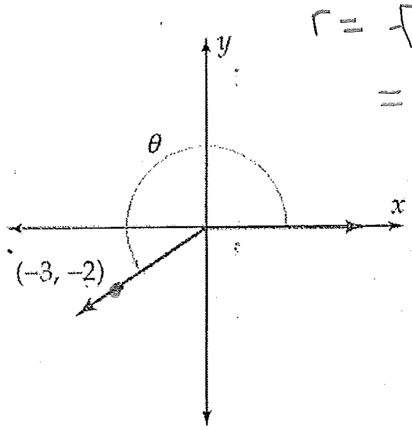
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



Example 1 Trigonometric Ratios in the Coordinate Plane

Find the sine, cosine, and tangent of the angle θ, whose terminal side passes through the point $(-3, -2)$.



$r = \sqrt{(-3)^2 + (-2)^2}$

$= \sqrt{13}$

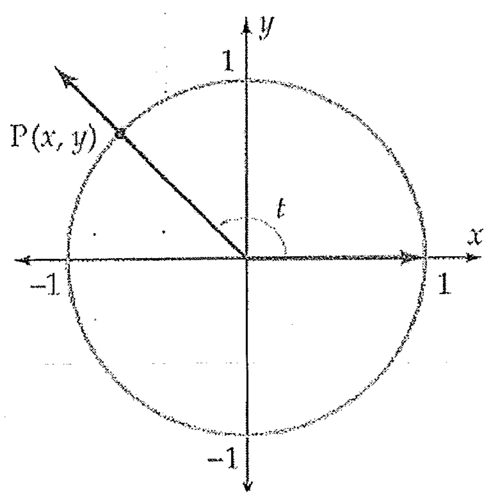
Sin Θ = $\frac{-2}{\sqrt{13}} = \frac{-2\sqrt{13}}{13}$

Cos Θ = $\frac{-3\sqrt{13}}{13}$

Tan Θ = $\frac{2}{3}$

Trigonometric Functions and the Unit Circle

Recall that the **unit circle** is the circle of radius 1 centered at the origin, whose equation is $x^2 + y^2 = 1$. The unit circle is the basis for the most useful description of trigonometric functions of real numbers.



$\sin t = \frac{y}{r}$ $\qquad \cos t = \frac{x}{r}$ $\qquad \tan t = \frac{y}{x}$

Sint = y $\qquad$ cost = x

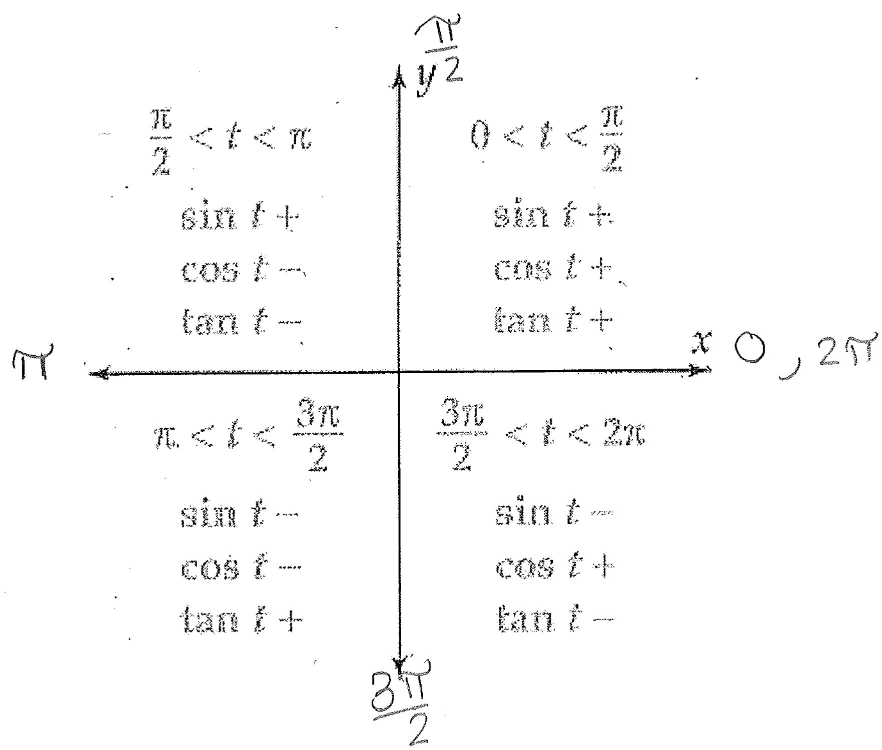
Domain and Range

> the domain of the sine function and of the cosine function is the set of all real numbers.
>
> the domain of the tangent function consists of all real numbers except $\pm\frac{\pi}{2} + 2k\pi$, where $k = 0, \pm1, \pm2, \pm3, \ldots$.

the range of the sine function and of the cosine function is the set of all real numbers between -1 and 1, that is, the interval $[-1, 1]$.

In contrast to sine and cosine,

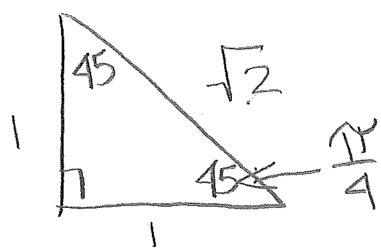
the range of the tangent function is the set of all real numbers.

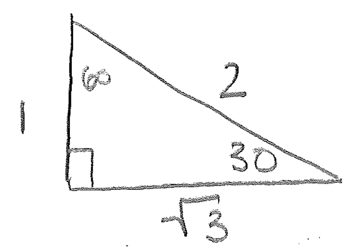


Exact Values of Trigonometric Functions

Although a calculator is used to evaluate trigonometric functions approximately, there are a few special numbers for which exact values can be found. Recall that 30°, 45°, and 60° are the same as $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$, respectively. Therefore, the chart on page 419 can be translated as follows.

t	30° $\frac{\pi}{6}$	45° $\frac{\pi}{4}$	60° $\frac{\pi}{3}$
sin t	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos t	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan t	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
csc t	$\frac{2}{1} = 2$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
sec t	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{1} = 2$
cot t	$\frac{\sqrt{3}}{1} = \sqrt{3}$	$\frac{1}{1} = 1$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$





The exact values of the trigonometric functions can also be found for any number that is an integer multiple of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$. The technique for doing this depends on the concept of a *reference angle*.

t	sin t	cos t	tan t
0	0	1	0
$\frac{\pi}{2}$	1	0	undefined
π	0	−1	0
$\frac{3\pi}{2}$	−1	0	undefined
2π	0	1	0

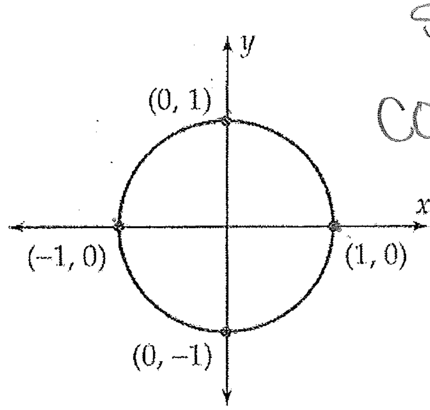


Figure 6.4-8

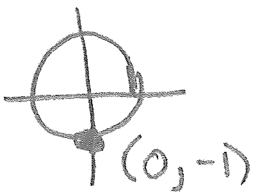
$\sin\frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1$

$\cos\frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0$

$\tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0} = \emptyset$

Example 2 Exact Values of Trigonometric Functions

Find the exact value of the sine, cosine, and tangent functions when $t = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π.



$\sin\frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1$

$\cos\frac{3\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0$

$\tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0} = \emptyset$

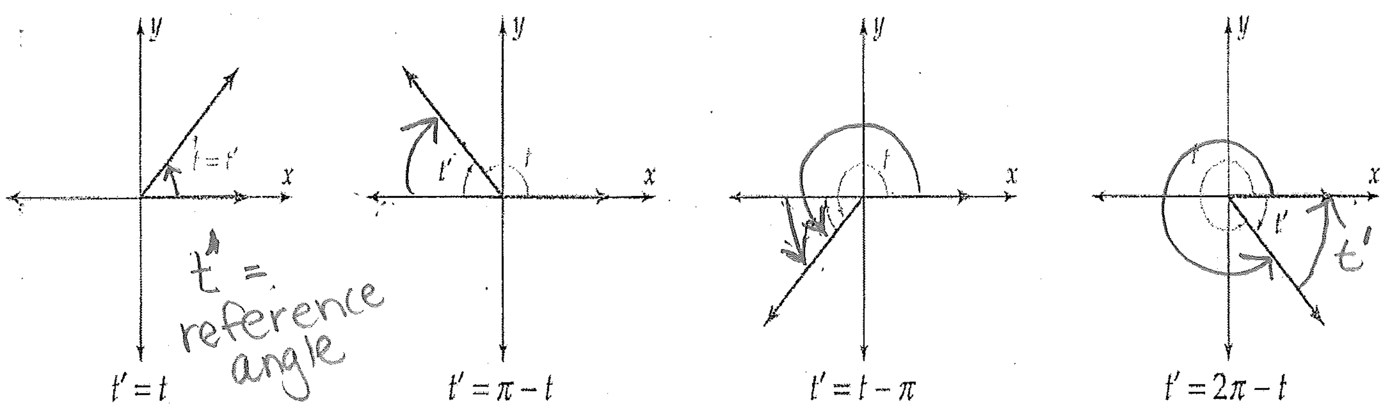
Reference Angles

Definition of Reference Angle

For an angle θ in standard position, the *reference angle* is the positive acute angle formed by the terminal side of θ and the x-axis.

Definition of Reference Angle

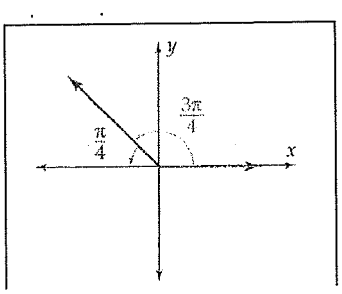
t

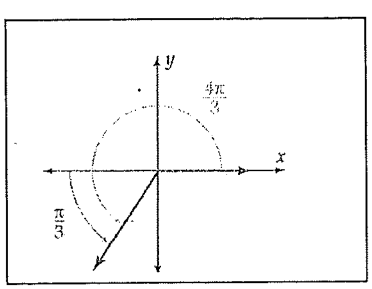


Example 3 Using Reference Angles

Use reference angles to find the exact value of $\sin t$, $\cos t$, and $\tan t$.

a. $t = \frac{3\pi}{4}$ **b.** $t = \frac{4\pi}{3}$ **c.** $t = \frac{11\pi}{6}$



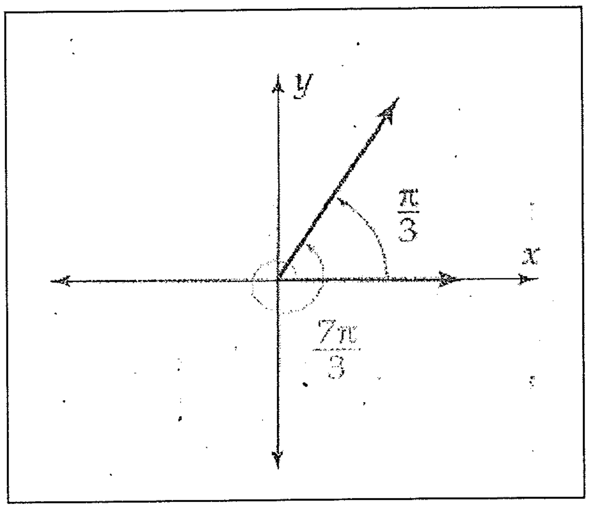


Trigonometric Ratios of Coterminal Angles

Any trigonometric function of a real number t is equal to the same trigonometric function of all numbers $t \pm 2k\pi$, where k is an integer.

Example 4 Trigonometric Functions Where $t > 2\pi$

Find the sine, cosine, and tangent of $\frac{7\pi}{3}$. $= 2\,\boxed{\frac{\pi}{3}}$ $60°$



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\tan\frac{\pi}{3} = \sqrt{3}$

Name_______________________________

Exercises 6.4

Note: Unless stated otherwise, all angles are in standard position. In exercises 1-6, find sin t, cos t and tan t, when the terminal side of an angle t radians passes through the

These are like example 1.

1. $(2, 7)$ **3.** $(-5, -6)$ **5.** $(\sqrt{3}, -10)$

In exercises 7-10, find sin t, cos t and tan t when the terminal side of an angle t radians passes through the given point on the UNIT CIRCLE. r=1

These are like example 1 except the radius is 1.

7. $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ **8.** $\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$ **9.** $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

In Exercises 11–14, identify an angle $0 \le t' \le \pi$ that is coterminal with the given angle, and find the sine and cosine of the given angle.

11. $\frac{13\pi}{6}$ **12.** $\frac{9\pi}{2}$ **13.** 16π **14.** $-\frac{7\pi}{4}$

In Exercises 15–23,

a. Use a calculator in radian mode to find the sine, cosine, and tangent of each number. Round your answers to four decimal places.

b. Use the signs of the functions to identify the quadrant of the terminal side of an angle of t radians. If the terminal side lies on an axis, identify which axis and whether it is on the positive or negative side of the axis. Explain your reasoning.

\

15. $\frac{7\pi}{5}$

16. 11

17. $-\frac{14\pi}{9}$

21. 9.5π

22. $\frac{\pi}{17}$

23. -17

In Exercises 24–29, sketch each angle whose radian measure is given and find its reference angle.

24. $\frac{7\pi}{3}$

25. $\frac{17\pi}{6}$

26. $\frac{6\pi}{5}$

27. 1.75π

28. $-\frac{3\pi}{4}$

29. $-\frac{\pi}{7}$

In Exercises 30–47, find the exact value of the sine, cosine, and tangent of the number without using a calculator.

30. $\frac{7\pi}{6}$ 31. $\frac{7\pi}{3}$ 32. $\frac{17\pi}{3}$ 33. $\frac{11\pi}{4}$ 34. $\frac{5\pi}{4}$ 35. $-\frac{3\pi}{2}$

36. 3π 37. $-\frac{23\pi}{6}$ 38. $\frac{11\pi}{6}$ 39. $-\frac{19\pi}{3}$ 40. $-\frac{10\pi}{3}$ 41. $-\frac{15\pi}{4}$

42. $-\frac{25\pi}{4}$ 43. $\frac{5\pi}{6}$ 44. $-\frac{17\pi}{2}$ 45. $\frac{9\pi}{2}$ 46. $-\pi$ 47. 4π

In Exercises 48–53, write the expression as a single real number. Do not use decimal approximations.

48. $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{2}\right)$

49. $\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$

51. $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{6}\right) - \cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{5\pi}{6}\right)$

53. $\sin\left(\frac{\pi}{3}\right)\cos\pi + \sin\pi\cos\left(\frac{\pi}{3}\right)$

In Exercises 54–59, the terminal side of an angle of t radians lies in the given quadrant on the given line. Find $\sin t$, $\cos t$, and $\tan t$. (*Hint:* Find a point on the terminal side of the angle.)

54. Quadrant III; line $2y - 4x = 0$

55. Quadrant IV; line through $(-3, 5)$ and $(-9, 15)$

56. Quadrant III; line through the origin parallel to $7x - 2y = -6$

59. Quadrant IV; line $y = -3x$