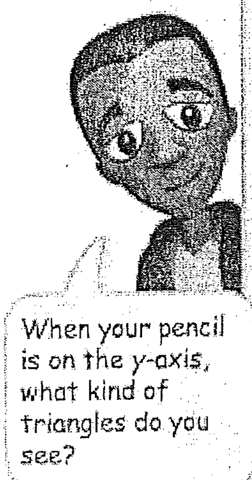
10-4 Ellipses

Content Standard

G.GPE.3 Derive the equations of ellipses . . . given foci, using the fact that the sum or difference of distances from the foci is constant.

Objectives To write the equation of an ellipse
To find the foci of an ellipse
To graph an ellipse

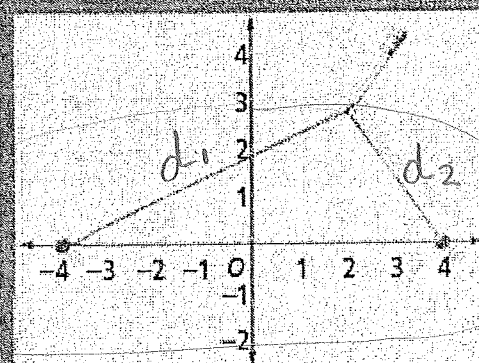


MATHEMATICAL PRACTICES

SOLVE IT! Getting Ready!

Suppose you have a piece of string 10 units long. You tack down its ends as shown. You place your pencil against the string and, keeping the string taut, you draw a smooth curve.

Where would your pencil hit each axis? Explain your reasoning.



Points on the smooth curve in the Solve It have a total distance of 10 units to the points $(-4, 0)$ and $(4, 0)$. In fact, all of the points on the smooth curve have a total distance of 10 units to the two fixed points. You can describe this smooth curve with an equation.

Essential Understanding A circle is the set of points a fixed distance from one point. An *ellipse* "stretches" a circle in one direction and is the set of points that have a total fixed distance from two points.

Take note

Key Concept Ellipse

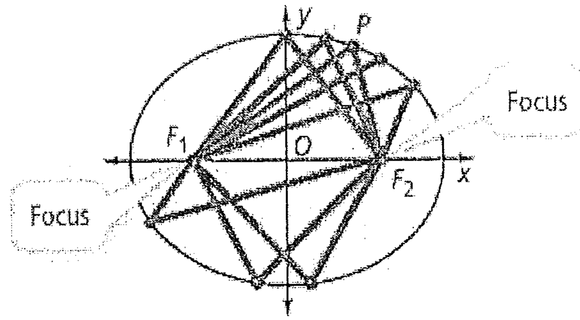
Definition

An **ellipse** is a set of all points P in a plane such that the sum of the distances from P to two fixed points F_1 and F_2 is a constant k. A **focus of an ellipse** (plural: foci) is one of the two fixed points.

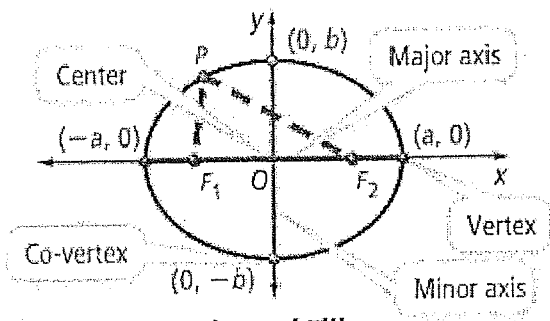
Symbols

$PF_1 + PF_2 = k$, where $k > F_1F_2$.

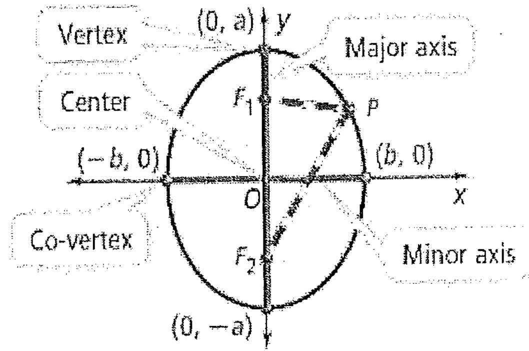
Graph



The **major axis** is the segment that contains the foci and has its endpoints on the ellipse. Its midpoint is the **center of the ellipse**. The **minor axis** is perpendicular to the major axis at the center. The **vertices of an ellipse** (singular: *vertex*) are the endpoints of the major axis. The **co-vertices of an ellipse** are the endpoints of the minor axis.



Horizontal Ellipse



Vertical Ellipse

Take note

Key Concept Properties of Ellipses with Center (0, 0)

	Horizontal Ellipses	Vertical Ellipses
Standard Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b > 0$
Major Axis	horizontal	vertical
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Co-vertices	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0)$ on x-axis	$(0, \pm c)$ on y-axis

The length of the major axis is $2a$ and the length of the minor axis is $2b$.
For any point P on an ellipse, $PF_1 + PF_2 = 2a$.

Problem 1 Writing an Equation of an Ellipse

What is an equation in standard form of an ellipse centered at the origin with vertex $(-6, 0)$ and co-vertex $(0, 3)$?

Think

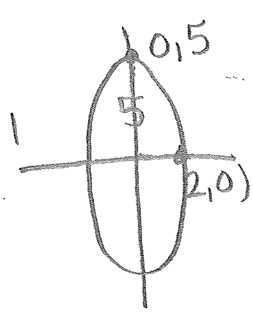
What is the orientation of the ellipse?
Since the vertices $(-6, 0)$ and $(6, 0)$ are aligned horizontally, the ellipse is horizontal.

Since one vertex is $(-6, 0)$, the other vertex is $(6, 0)$. The major axis is horizontal.
Since one co-vertex is $(0, 3)$, the other co-vertex is $(0, -3)$. The minor axis is vertical.
So $a = 6$, $b = 3$, $a^2 = 36$, and $b^2 = 9$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Standard form of a horizontal ellipse

$\frac{x^2}{36} + \frac{y^2}{9} = 1$ Substitute for a^2 and b^2.

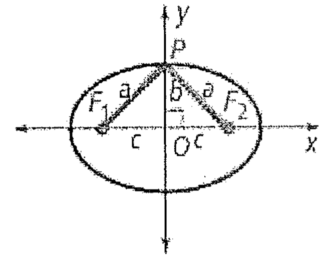
$\frac{x^2}{4} + \frac{y^2}{25} = 1$



b=2
a=5

Got It? 1. What is the equation in standard form of an ellipse centered at the origin with vertex $(0, 5)$ and co-vertex $(2, 0)$?

Since the co-vertex $P(0, b)$ is on the ellipse, $PF_1 + PF_2 = 2a$. If you denote the distance from each focus to the center of the ellipse by c, then a, b, and c are the lengths of the sides of a right triangle, as shown in the ellipse at the right. Thus, the distances from the center to each vertex, to each co-vertex, and to each focus are related by the Pythagorean Theorem: $a^2 = b^2 + c^2$.



If $(\pm a, 0)$, $(0, \pm b)$, and $(\pm c, 0)$ are the vertices, the co-vertices, and the foci of an ellipse, respectively,

$$c^2 = a^2 - b^2$$

Problem 2 Finding the Foci of an Ellipse

What are the foci of the ellipse with the equation $25x^2 + 9y^2 = 225$? Graph the foci and the ellipse.

Know	Need	Plan
The equation of an ellipse.	The coordinates of the vertices, co-vertices, and foci.	• Write the equation in standard form to find a^2 and b^2. Use $c^2 = a^2 - b^2$ to find c. • Use a, b, and c to graph the ellipse.

$$25x^2 + 9y^2 = 225$$

$$\frac{25x^2}{225} + \frac{9y^2}{225} = 1 \qquad \text{Divide each side by 225.}$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \text{Simplify to standard form.}$$

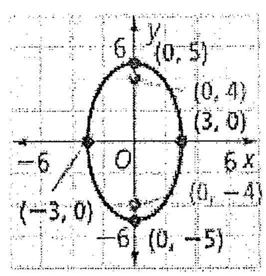
Since $25 > 9$ and 25 is with y^2, the major axis is vertical.

$a^2 = 25$ and $b^2 = 9$

$c^2 = a^2 - b^2 = 25 - 9 = 16$

$c = \pm 4$

The foci are $(0, 4)$ and $(0, -4)$. Plot points for the vertices, co-vertices, and foci, then graph the ellipse.



"c"

Got It? 2. a. What are the coordinates of the foci of the ellipse with the equation $36x^2 + 100y^2 = 3600$? Graph the ellipse.

$$\frac{36x^2}{3600} + \frac{100y^2}{3600} = \frac{3600}{3600}$$

$$\frac{x^2}{\frac{3600}{36}} + \frac{y^2}{\frac{3600}{100}} = 1$$

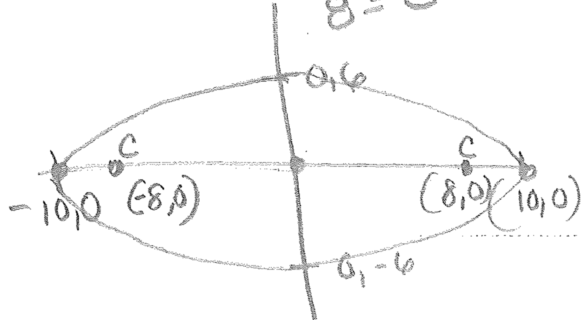
$$\frac{x^2}{100} + \frac{y^2}{36}$$

$a = 10$ $\quad b = 6$

$100 - 36 = c^2$

$64 = c^2$

$8 = c$



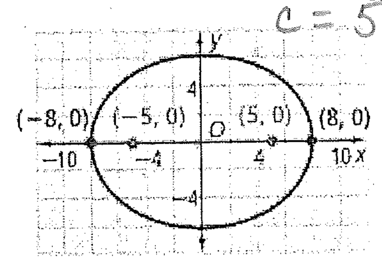
$a^2 - b^2 = c^2$
$64 - b^2 = 25$
$b = ?$ $a = 8$
$c = 5$

Problem 4 Using the Foci of an Ellipse

Think

How can you write the equation of an ellipse given a focus and a vertex?
Find the values of a and c. Use a^2 and c^2 to find b^2.

What is the standard form equation of the ellipse shown?



The foci are on the x-axis, so the major axis is horizontal.

Since $c = 5$ and $a = 8$, $c^2 = 25$ and $a^2 = 64$.

$c^2 = a^2 - b^2$

$25 = 64 - b^2$ Substitute.

$b^2 = 39$ Solve.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Standard form of a horizontal ellipse

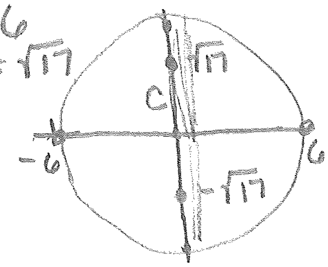
$\frac{x^2}{64} + \frac{y^2}{39} = 1$ Substitute.

$\frac{x^2}{64} + \frac{y^2}{b^2} = 1$
$\frac{x^2}{64} + \frac{y^2}{39} = 1$

Got It? **4.** What is the standard form equation of an ellipse with foci at $(0, \pm\sqrt{17})$ and co-vertices at $(\pm 6, 0)$?

$a^2 - b^2 = c^2$
$a^2 - 36 = 17$
$a^2 = 43$

$b = 6$
$c = \sqrt{17}$



$\frac{x^2}{36} + \frac{y^2}{43} = 1$

Practice and Problem-Solving Exercises

Write an equation of an ellipse in standard form with center at the origin and with the given vertex and co-vertex listed respectively. See Problem 1.

7. (4, 0), (0, 3) **8.** (0, 1), (2, 0) **9.** (3, 0), (0, −1) **10.** (0, 6), (1, 0)

Find the foci for each equation of an ellipse. Then graph the ellipse. See Problem 2.

15. $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **16.** $\frac{x^2}{9} + \frac{y^2}{25} = 1$ **17.** $\frac{x^2}{81} + \frac{y^2}{49} = 1$ **18.** $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Find the distance between the foci of an ellipse. The lengths of the major and minor axes are listed respectively. See Problem 3.

23. 40 and 24 **24.** 30 and 18 **25.** 10 and 8 **26.** 16 and 10

Write an equation of an ellipse for the given foci and co-vertices. See Problem 4.

31. foci $(\pm 6, 0)$, co-vertices $(0, \pm 8)$

32. foci $(0, \pm 8)$, co-vertices $(\pm 8, 0)$

Write an equation of an ellipse for the given foci and co-vertices. See Problem 4.

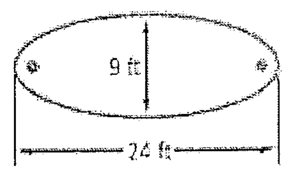
31. foci $(\pm 6, 0)$, co-vertices $(0, \pm 8)$

32. foci $(0, \pm 8)$, co-vertices $(\pm 8, 0)$

35. **Miniature Golf** The figure at the right represents a miniature golf green. The green is elliptical with the tee at one focus and the hole at the other.

a. How far is the hole from the tee?

b. Knowing that the border is elliptical, how should you aim your putt from the tee?



Find the foci for each equation of an ellipse.

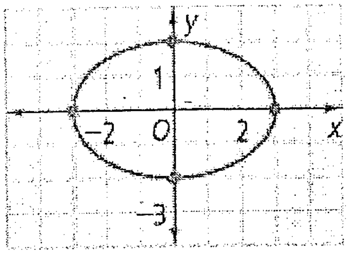
36. $4x^2 + 9y^2 = 36$

37. $16x^2 + 4y^2 = 64$

38. $4x^2 + 36y^2 = 144$

Write an equation for each ellipse.

44.



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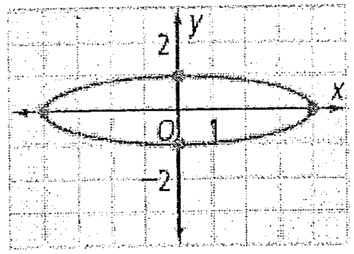

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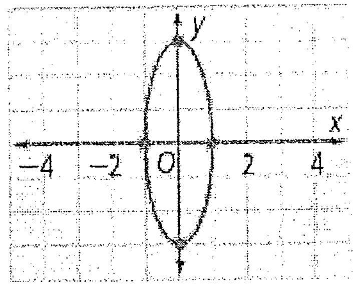

Write an equation of an ellipse in standard form with center at the origin and with the given characteristics.

48. focus $(1, 0)$, width 4

49. $a = 5$, $b = 2$, width 10

50. vertex $(-11, 0)$, co-vertex $(0, 9)$

51. height 29, width 53

52. focus $(-5, 0)$, co-vertex $(0, -12)$

53. $c^2 = 68$, vertex $(0, -18)$

54. focus $(0, 3\sqrt{2})$, height 19

55. focus $(2, 0)$, x-intercept 4