11-1 Permutations and Combinations

Content Standard

S.CP.9 Use permutations and combinations to compute probabilities of compound events and solve problems.

Objectives To count permutations
To count combinations

Probability

BIG idea Probability expresses the likelihood that a particular event will occur. Data can be used to calculate an experimental probability, and mathematical properties can be used to determine a theoretical probability. Either experimental or theoretical probability can be used to make predictions or decisions about future events. Various counting methods can be used to develop theoretical probabilities.

SOLVE IT! Getting Ready!

For lunch, in how many different ways could you choose a sandwich, side dish, and dessert? Explain your reasoning.

Cafeteria Menu

Sandwich	Side	Dessert
Hamburger	Potatoes	Apple Crisp
Cheeseburger	Beans	Banana
Veggieburger	Corn	Flan
PB&J		Rice Pudding

Try starting with your favorite sandwich and find out how many choices you have.

MATHEMATICAL PRACTICES

Lesson Vocabulary
- Fundamental Counting Principle
- permutation
- *n* factorial
- combination

It is fairly easy to count the ways you can pick items from a short list. But, sometimes you have so many choices that counting the possibilities is impractical.

Essential Understanding You can use multiplication to quickly count the number of ways certain things can happen.

The **Fundamental Counting Principle** describes the method of using multiplication to count.

take note **Key Concept** **Fundamental Counting Principle**

If event M can occur in m ways and is followed by event N that can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

Example 3 pants and 2 shirts give $3 \cdot 2 = 6$ possible outfits.

Problem 1 Using the Fundamental Counting Principle

Motor Vehicles The photos show Maryland license plates in 2004 and 1912. How many more 2004-style license plates were possible than 1912-style plates?

Think

How many digits are in our number system? How many letters are in our alphabet?
There are 10 digits and 26 letters.

For the 2004 license plates, there were places for three letters and three digits. Number of possible 2004 license plates:

$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$

For the 1912 license plates, there were places for four digits. Number of possible 1912 license plates:

$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$

$17{,}576{,}000 - 10{,}000 = 17{,}566{,}000$ Find the difference.

There were 17,566,000 more 2004-style license plates possible than 1912-style plates.

Got It? 1. In 1966, one type of Maryland license plate had two letters followed by four digits. How many of this type of license plate were possible?

A **permutation** is an arrangement of items in a particular order. Suppose you wanted to find the number of ways to order three items. There are 3 ways to choose the first item, 2 ways to choose the second, and 1 way to choose the third. By the Fundamental Counting Principle, there are $3 \cdot 2 \cdot 1 = 6$ permutations.

Using *factorial* notation, you can write $3 \cdot 2 \cdot 1$ as 3!, read "three factorial." For any positive integer n, **n factorial** is $n! = n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. Zero factorial is $0! = 1$.

Problem 2 Finding the Number of Permutations of *n* Items

Plan

What strategy can you use to help you determine the answer?
Act it out and you will see how many options you have at each step.

In how many ways can you file 12 folders, one after another, in a drawer?

Use the Fundamental Counting Principle to count the number of permutations of 12 items. There are 12 ways to select the first folder, 11 ways to select the next folder, and so on. The total number of permutations is

$12! = 12 \cdot 11 \cdot \ldots \cdot 2 \cdot 1 = 479{,}001{,}600.$

There are 479,001,600 ways to file 12 folders in a drawer.

Got It? 2. In how many ways can you arrange 8 shirts on hangers in a closet?

Problem 5 Identifying Whether Order Is Important

For each situation, determine whether you should use a permutation or combination. What is the answer to each question?

A **A chemistry teacher divides his class into eight groups. Each group submits one drawing of the molecular structure of water. He will select four of the drawings to display. In how many different ways can he select the drawings?**

There is no reason why order is important. Use a combination.

$${}_nC_r = \frac{n!}{r!(n-r)!}; \qquad {}_8C_4 = \frac{8!}{4!(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{4!}} = 70$$

There are 70 ways to select the drawings.

B **You will draw winners from a total of 25 tickets in a raffle. The first ticket wins \$100. The second ticket wins \$50. The third ticket wins \$10. In how many different ways can you draw the three winning tickets?**

Values of the tickets depend on the order in which you draw them. Order is important. Use a permutation.

$${}_nP_r = \frac{n!}{(n-r)!}; \qquad {}_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = 25 \cdot 24 \cdot 23 = 13{,}800$$

There are 13,800 ways you can draw the winning tickets.

Got It? **5.** In Problem 5A, how many ways are possible for the teacher to select and arrange the four drawings from left to right on the wall?

9. You have five shirts and four pairs of pants. How many different ways can you arrange your shirts and pants into outfits? See Problem 1.

10. To create an entry code for a push-button door lock, you need to first choose a letter and then, three single-digit numbers. How many different entry codes can you create?

11. The prom committee has four sites available for the banquet and three sites for the dance. How many arrangements are possible for the banquet and dance?

Evaluate each expression. See Problem 2.

13. $10!$

15. $5!3!$

17. $5(4!)$

19. $\frac{15!}{10!5!}$

Evaluate each expression. See Problem 3.

21. ${}_8P_1$

23. ${}_8P_3$

25. ${}_3P_2$

27. ${}_9P_6$

29. **Scheduling** Fifteen students ask to visit a college admissions counselor. Each scheduled visit includes one student. In how many ways can ten time slots be assigned?

Evaluate each expression. See Problem 4.

31. ${}_8C_5$

32. ${}_4C_4$

33. ${}_4C_3$

35. $3({}_5C_4)$

37. $\frac{{}_7C_4}{{}_9C_4}$

38. **Awards** There are eight swimmers in a competition where the top three swimmers advance. In how many ways can three swimmers advance?

For each situation, determine whether to use a permutation or a combination. Then solve the problem. See Problem 5.

39. How many different teams of 11 players can be chosen from a soccer team of 16?

40. Suppose you find seven equally useful articles related to the topic of your research paper. In how many ways can you choose five articles to read?

41. A salad bar offers eight choices of toppings for a salad. In how many ways can you choose four toppings?

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$4 \cdot 3 \cdot 4 = 48$

MATHEMATICAL PRACTICES

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$\underset{\text{letter}}{\underline{26}} \cdot \underset{\text{letter}}{\underline{26}} \cdot \underset{\text{digit}}{\underline{10}} \cdot \underset{\text{digit}}{\underline{10}} \cdot \underset{\text{digit}}{\underline{10}} \cdot \underset{\text{digit}}{\underline{10}}$

6,760,000

A **permutation** is an arrangement of items in a particular order. Suppose you wanted to find the number of ways to order three items. There are 3 ways to choose the first item, 2 ways to choose the second, and 1 way to choose the third. By the Fundamental Counting Principle, there are $3 \cdot 2 \cdot 1 = 6$ permutations.

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Problem 2 Finding the Number of Permutations of n Items

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$12! = 12 \cdot 11 \cdot \ldots \cdot 2 \cdot 1 = 479{,}001{,}600.$

There are 479,001,600 ways to file 12 folders in a drawer.

12!

$12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

Got It? 2. In how many ways can you arrange 8 shirts on hangers in a closet?

8!

$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

40,320

$_nP_r$ $\quad {}_{10}P_3 \quad \frac{n!}{(n-r)!} = \frac{10!}{(10-3)!}$

$\frac{10!}{7!}$

take note

Key Concept Number of Permutations

The number of permutations of n items of a set arranged r items at a time is

$${}_nP_r = \frac{n!}{(n-r)!} \text{ for } 0 \leq r \leq n.$$

Example ${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$

$\frac{10 \cdot 9 \cdot 8 \cdot \cancel{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}$

$10 \cdot 9 \cdot 8$

720

Problem 3 Finding ${}_nP_r$

Track Ten students are in a race. First, second, and third places will win medals. In how many ways can 10 runners finish first, second, and third (no ties allowed)?

Plan

If you use the Fundamental Counting Principle, how many numbers will you need to multiply together? You will multiply 3 numbers together to represent the possibilities of finishing first, second, and third.

Method 1 Use the Fundamental Counting Principle.

$10 \cdot 9 \cdot 8 = 720$

Method 2 Use the permutation formula.

There are $n = 10$ runners to arrange taking $r = 3$ at a time.

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{Use the formula.}$$

$$= \frac{10!}{(10-3)!} \quad \text{Substitute 10 for } n \text{ and 3 for } r.$$

$$= \frac{10!}{7!} = 720 \quad \text{Simplify.}$$

There are 720 ways that 10 runners can finish in first, second, and third places.

Got It? 3. a. In how many ways can 15 runners finish first, second, and third?

b. **Reasoning** In Problem 3, is the number of ways for runners to finish first, second, and third the same as the number of ways to finish eighth, ninth, and tenth? Explain.

${}_{15}P_3$

$\frac{15!}{(15-3)!} = \frac{15!}{12!}$

$\frac{15 \cdot 14 \cdot 13 \cdot \cancel{12 \cdot 11 \cdot 10}}{\cancel{12 \cdot 11 \cdot 10}}$

2730

Suppose in Problem 3 that the first three runners advance to the championship race. In that case, the order in which the first three runners cross the finish line does not matter. A selection in which order does not matter is a **combination**.

As with permutations, you can use a formula to find the number of combinations of n items chosen r at a time.

Take note

Key Concept Number of Combinations

The number of combinations of n items of a set chosen r items at a time is

$$_nC_r = \frac{n!}{r!(n-r)!} \text{ for } 0 \leq r \leq n.$$

Example $_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \cdot 2!} = \frac{120}{6 \cdot 2} = 10$

$$\frac{5!}{3!(5-3)!}$$

$$\frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1} \cdot \cancel{2} \cdot 1}$$

10

$_5C_3$

Problem 4 Finding $_nC_r$

What is $_{13}C_4$, the number of combinations of 13 items taken 4 at a time?

Think	Write
You need the formula for number of combinations. Substitute 13 for n and 4 for r.	$_nC_r = \frac{n!}{r!(n-r)!}$ $_{13}C_4 = \frac{13!}{4!(13-4)!}$ $= \frac{13!}{4! \cdot 9!}$
Write out the factorial in the numerator to make it easier to divide. Remove common factors.	$= \frac{13 \cdot \cancel{12} \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot \cancel{9!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{9!}} = 715$

Got It? 4. What is the value of each expression?

a. $_8C_3$ **b.** $_9C_2$ **c.** $_{15}C_5$

a. $\frac{8!}{3!\,5!}$

$$\frac{8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{3 \cdot 2 \cdot 1 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}$$

56

b. $\frac{9!}{2!\,7!}$

36

c. 3003

$\frac{15!}{5!(10!)}$

When determining whether to use a permutation or combination, you must decide whether order is important.

Problem 5 Identifying Whether Order Is Important

For each situation, determine whether you should use a permutation or combination. What is the answer to each question?

A A chemistry teacher divides his class into eight groups. Each group submits one drawing of the molecular structure of water. He will select four of the drawings to display. In how many different ways can he select the drawings?

There is no reason why order is important. Use a combination.

$$_nC_r = \frac{n!}{r!(n-r)!}; \qquad {}_8C_4 = \frac{8!}{4!(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{4!}} = 70$$

There are 70 ways to select the drawings.

B You will draw winners from a total of 25 tickets in a raffle. The first ticket wins \$100. The second ticket wins \$50. The third ticket wins \$10. In how many different ways can you draw the three winning tickets?

Values of the tickets depend on the order in which you draw them. Order is important. Use a permutation.

$$_nP_r = \frac{n!}{(n-r)!}; \qquad {}_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = 25 \cdot 24 \cdot 23 = 13{,}800$$

There are 13,800 ways you can draw the winning tickets.

Got It? 5. In Problem 5A, how many ways are possible for the teacher to select and arrange the four drawings from left to right on the wall?

$_8P_4$ $\qquad \frac{8!}{4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}$

$= 1680$

9. You have five shirts and four pairs of pants. How many different ways can you arrange your shirts and pants into outfits? See Problem 1.

$5 \cdot 4 = 20$

10. To create an entry code for a push-button door lock, you need to first choose a letter and then, three single-digit numbers. How many different entry codes can you create?

$\underset{\text{letter}}{\underline{26}}\ \underset{\text{digit}}{\underline{10}}\ \underset{\text{digit}}{\underline{10}}\ \underset{\text{digit}}{\underline{10}} = 26{,}000$

11. The prom committee has four sites available for the banquet and three sites for the dance. How many arrangements are possible for the banquet and dance?

$4 \cdot 3 = 12$

Evaluate each expression. See Problem 2.

13. 10!

3,628,800

15. 5!3!

720

17. 5(4!)

120

19. $\frac{15!}{10!5!}$

3003

Evaluate each expression. $\frac{n!}{(n-r)!}$ See Problem 3.

21. ${}_8P_1$

$\frac{8!}{7!} = 8$

23. ${}_8P_3$

$\frac{8!}{5!} = 336$

25. ${}_3P_2$

$\frac{3!}{1!} = 6$

27. ${}_9P_6$

$\frac{9!}{3!} =$ ~~[illegible]~~

60,480

29. Scheduling Fifteen students ask to visit a college admissions counselor. Each scheduled visit includes one student. In how many ways can ten time slots be assigned?

${}_{15}P_{10}$ $\frac{15!}{5!}$ 10,897,286,400

Evaluate each expression. See Problem 4.

31. ${}_8C_5$

$\frac{8!}{5!\,3!} = 56$

32. ${}_4C_4$

$\frac{4!}{4!\,0!}$

1

33. ${}_4C_3$

$\frac{4!}{3!\,1!} = 4$

35. $3({}_5C_4)$

15

37. $\frac{{}_7C_4}{{}_9C_4}$

$\frac{5}{18}$

38. Awards There are eight swimmers in a competition where the top three swimmers advance. In how many ways can three swimmers advance?

${}_8C_3 = 56$

$\frac{8!}{3!\,5!}$

For each situation, determine whether to use a permutation or a combination. Then solve the problem. See Problem 5.

39. How many different teams of 11 players can be chosen from a soccer team of 16? ${}_{16}C_{11}$ 4368

Combination

40. Suppose you find seven equally useful articles related to the topic of your research paper. In how many ways can you choose five articles to read? Combination ${}_7C_5$ 21

41. A salad bar offers eight choices of toppings for a salad. In how many ways can you choose four toppings? Combination

${}_8C_4$ 70