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4.5 Complex Numbers

Complex Numbers

Equations such as $x^2 = -1$ and $x^2 = -4$ have no solutions in the real number system because $\sqrt{-1}$ and $\sqrt{-4}$ are not real numbers. In order to solve such equations, that is, to find the square roots of negative numbers, the number system must be enlarged again. There is a number system, called the complex number system, with the desired properties.

$$\sqrt{-1} = i$$
$$(\sqrt{-1})^2 = i^2$$
$$-1 = i^2$$

So always replace $i^2 = -1$

Solve: $x^2 + 4x + 5 = 0$

$$x^2 + 4x + 4 = -5 + 4$$
$$\sqrt{(x+2)^2} = \sqrt{-1}$$
$$x + 2 = \pm i$$
$$x = \underbrace{-2}_{\text{real}} + \underbrace{\pm i}_{\text{imaginary}}$$

Complex

Numbers of the form bi, where b is a real number, are called **imaginary numbers.** Sums of real and imaginary numbers, numbers of the form $a + bi$, are called **complex numbers.** For example,

$$5 + 2i \qquad 7 - 4i \qquad 18 + \frac{3}{2}i \qquad 3 - 12i \longrightarrow$$

are all complex numbers.

Standard form of a complex number is : $a + bi$

Real number + *Imaginary number*

Example 1 Equating Two Complex Numbers

Find x and y if $\overset{a}{2x} - \overset{bi}{3i} = \overset{a}{-6} + \overset{bi}{4yi}$.

$a = a$
$2x = -6$
$x = -3$

$bi = bi$
$-3\cancel{i} = 4y\cancel{i}$

$\frac{-3}{4} = \frac{4y}{4}$
$y = -3/4$

In Exercises 55–58, find x and y.

55. $3x - 4i = 6 + 2yi$

56. $8 - 2yi = 4x + 12i$

57. $3 + 4xi = 2y - 3i$

58. $8 - xi = \frac{1}{2}y + 2i$

Arithmetic of Complex Numbers

Because the usual laws of arithmetic hold, it is easy to add, subtract, and multiply complex numbers. As the following examples demonstrate,

all symbols can be treated as if they were real numbers, provided that i^2 is replaced by -1.

Example 2 Adding, Subtracting, and Multiplying Complex Numbers

$i^2 = -1$

Perform the indicated operation and write the result in the form $a + bi$.

a. $(1 + i) + (3 - 7i)$

$4 - 6i$

b. $(4 + 3i) - (8 - 6i)$

$4 + 3i - 8 + 6i$

$-4 + 9i$

c. $4i\left(2 + \frac{1}{2}i\right)$

$8i + 2i^2$

$-2 + 8i$

d. $(2 + i)(3 - 4i)$

a. $(3 + 2i)(3 - 2i)$

$9 \cancel{-6i} \cancel{+6i} - 4i^2 \quad {+4}$

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b. $(4 + i)^2$

$(4+i)(4+i)$

$16 + 8i + i^2$

$15 + 8i$

In Exercises 1–54, perform the indicated operation and write the result in the form $a + bi$.

1. $(2 + 3i) + (6 - i)$

3. $(2 - 8i) - (4 + 2i)$

5. $\frac{5}{4} - \left(\frac{7}{4} + 2i\right)$

7. $\left(\frac{\sqrt{2}}{2} + i\right) - \left(\frac{\sqrt{3}}{2} - i\right)$

9. $(2 + i)(3 + 5i)$

11. $(-3 + 2i)(4 - i)$

13. $(2 - 5i)^2$

15. $(\sqrt{3} + i)(\sqrt{3} - i)$

$\boxed{i^2 = -1}$

Powers of i

Observe that

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

The powers of i form a cycle. Any power of i must be one of four values: 1, -1, i or $-i$.

Example 4 Powers of i

Find i^{54}.

$(\cancel{i^2})^{27}$

$(-1)^{27} = -1$

$i^{55} = (\cancel{i^2})^{27} i$

$(-1)^{27} i$

$-i$

17. i^{15}

18. i^{26}

19. i^{53}

20. $(-i)^{55}$

21. $(-i)^{107}$

22. $(-i)^{213}$

Example 5 Quotients of Two Complex Numbers

Express the quotient $\frac{3+4i}{1+2i}$ in standard form.

$\frac{(3+4i)(1-2i)}{(1+2i)(1-2i)} = \frac{3-6i+4i-8i^2}{1-4i^2}$

$\frac{11-2i}{5}$

Complex Conjugates

The conjugate of the complex number $a + bi$ is the number $a - bi$, and the conjugate of $a - bi$ is $a + bi$. For example, the conjugate of $3 + 4i$ is $3 - 4i$, and the conjugate of $-3i = 0 - 3i$ is $0 + 3i = 3i$. The numbers $3 + 4i$ and $3 - 4i$ are called **conjugate pairs**. Because $a + 0i = a - 0i$ for each real number a, *every real number is its own conjugate.*

$\frac{1}{i(4+5i)}$

23. $\frac{1}{5-2i}$ $\frac{(5+2i)}{(5+2i)}$

25. $\frac{1}{3i}$ $\frac{(-3i)}{(-3i)}$

27. $\frac{3}{4+5i}$

29. $\frac{1}{i(4+5i)} = \frac{1}{4i-5}$

$\frac{1}{(-5+4i)} \frac{(-5-4i)}{(-5-4i)}$

31. $\frac{2+3i}{i(4+i)}$

33. $\frac{2+i}{1-i} + \frac{1}{1+2i}$

35. $\frac{i}{3+i} - \frac{3+i}{4+i}$

Example 6 Square Roots of Negative Numbers

$\sqrt{-4}$
$\sqrt{-1}\cdot\sqrt{4}$
$i\cdot 2$

$\sqrt{-9}$
$\sqrt{-1}\cdot\sqrt{9}$
$i\cdot 3$
$3i$

Write each of the following as a complex number.

a. $\sqrt{-3}$

$\sqrt{3}\,i$

b. $\frac{1-\sqrt{-7}}{3}$

$\frac{1-\sqrt{7}i}{3}$

c. $(7-\sqrt{-4})(5+\sqrt{-9})$

$(7-2i)(5+3i)$

$35+21i-10i-6i^2 = \boxed{41+11i}$

37. $\sqrt{-36}$

39. $\sqrt{-14}$

41. $-\sqrt{-16}$

43. $\sqrt{-16}+\sqrt{-49}$

45. $\sqrt{-15}-\sqrt{-18}$

47. $\frac{\sqrt{-16}}{\sqrt{-36}}$

49. $(\sqrt{-25}+2)(\sqrt{-49}-3)$

51. $(2+\sqrt{-5})(1-\sqrt{-10})$

53. $\frac{1}{1+\sqrt{-2}}$

Every quadratic equation with real coefficients has solutions in the complex number system.

ALL (real + imaginary)

Conjugate Solutions: If a + bi is a solution of a polynomial equation with real coefficients, then its conjugate, a –bi, is also a solution of the equation.

Example 7 Complex Solutions to a Quadratic Equation

Find all solutions to $2x^2 + x + 3 = 0$.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\frac{-1 \pm \sqrt{1-4(2)(3)}}{4}$$

$$\frac{-1 \pm \sqrt{-23}}{4} \Rightarrow \frac{-1 \pm \sqrt{23}\,i}{4}$$

59. $3x^2 - 2x + 5 = 0$ 61. $x^2 + x + 2 = 0$ 63. $2x^2 - x = -4$ 65. $2x^2 + 3 = 6x$

Example 8 Zeros of Unity

Find all solutions of $x^3 = 1$.

67. $x^3 - 8 = 0$

69. $x^4 - 1 = 0$

$$x^3 - 1 = 0$$

x x x 1 1

$$(x-1)(x^2 + 1x + 1)$$

$$x = 1$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\frac{-1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$\frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$$