4.6 Fundamental Theorem of Algebra Name______________________

Fundamental Theorem of Algebra: Every non constant polynomial has a zero in the complex number system.

Example: 5 = 5 + 0i

The goal of this section is to find **all** zeros of the polynomial (**real and imaginary**). Keep in mind that all imaginary zeros come in **complex conjugate pairs**. So if 2 + 3i is a zero, then so is 2- 3i

Also remember that when you are finding a polynomial for a set of zeros, there can be more than one answer.

EXAMPLE 1: Find "a" polynomial for the given conditions….
Degree is 4, zeros are 3i and -3i with a multiplicity of 2 for each zero.

$$f(x) = a(x-(3i))^2(x-(-3i))^2$$

$$(x-3i)^2(x+3i)^2$$

switch to * these together to get (x^2+9)

$$(x-3i)(x-3i)(x+3i)(x+3i)$$

$$(x-3i)(x+3i)(x-3i)(x+3i)$$

$$(x^2+9)(x^2+9)$$

$$x^4+9x^2+9x^2+81$$

$$f(x) = x^4+18x^2+81$$

EXAMPLE 2: Find a polynomial with a degree of 3 with zeros at -1, ½, 2, where f(0) =2. There is only one answer!

x y

find a

$$f(x) = a(x-(-1))(x-(\tfrac{1}{2}))(x-(2))$$

$$f(x) = a(x+1)(x-\tfrac{1}{2})(x-2)$$

$$2 = a(0+1)(0-\tfrac{1}{2})(0-2) \leftarrow \text{plug 0 in for } x$$

$$2 = a(1)(-\tfrac{1}{2})(-2)$$

$$2 = a$$

$$f(x) = 2(x+1)(x-\tfrac{1}{2})(x-2)$$

"a" ← doesn't matter what a value is

EXAMPLE 3: Find a polynomial with a degree of 2 and one of the zeros is 1+3i. ← can't have w/o $1-3i$

$$f(x) = (x-(1+3i))(x-(1-3i))$$
$$((x-1)-3i)((x-1)+3i)$$
$$(x-1)^2+9$$
$$x^2-2x+1+9$$
$$f(x) = x^2-2x+10$$

EXAMPLE 4: Find a polynomial with a degree of 2 with zero at 1+3i and f(2)= 3. $(1-3i)$

$$f(x) = a(x^2-2x+10)$$
$$3 = a(2^2-2(2)+10) \leftarrow$$ plug in 2 for x & 3 for y
$$3 = 10a$$
$$3/10 = a$$
✱ can leave like this
$$f(x) = \frac{3}{10}(x^2-2x+10) = \frac{3}{10}x^2 - \frac{3}{5}x + 3$$

EXAMPLE 5: Find all zeros of $x^4-5x^3+10x^2-20x+24$ if 2i is one of the zeros. $(x-2i)(x+2i)$

$\Downarrow$ $-2i$ $\Rightarrow x^2+4$

$$\begin{array}{r} x^2-5x+6 \\ x^2+0x+4 \,\big)\, \overline{x^4-5x^3+10x^2-20x+24} \\ -(x^4+0x^3+4x^2) \\ \hline -5x^3+6x^2-20x \\ -(-5x^3+0x^2-20x) \\ \hline 6x^2+0x+24 \\ -(6x^2+0x+24) \\ \hline 0 \end{array}$$

$$(x^2+4)(x^2-5x+6)$$
$$(x^2+4)(x-3)(x-2)$$
$$x=\pm 2i \qquad x=3 \qquad x=2$$

EXAMPLE 6: Find all the zeros: $2x^4-7x^2-4=0$

$$(2x^2+1)(x^2-4)$$

$$2x^2+1=0 \qquad x^2-4=0$$
$$\sqrt{x^2}=\sqrt{-\tfrac{1}{2}} \qquad x^2=4$$
$$x=\pm\sqrt{\tfrac{1}{2}}\,i \qquad x=\pm 2$$
$$x=\pm\frac{\sqrt{2}}{2}i$$

Remember to use the Factor Theorem to determine if something is a factor.

EXAMPLE 7: Is $x-2$ a factor of x^6-10?

$$2^6-10$$
$$64-10=54 \leftarrow$$ NO, need remainder of 0

Exercises 4.6

In Exercises 1–6, determine if $g(x)$ is a factor of $f(x)$ without using synthetic or long division.

Use the factor theorem or remainder theorem.

1. $f(x) = x^{10} + x^{9} \qquad g(x) = x - 1$

3. $f(x) = 3x^4 - 6x^3 + 2x - 1 \qquad g(x) = x + 1$

5. $f(x) = x^3 - 2x^2 + 5x - 4 \qquad g(x) = x + 2$

In Exercises 7–10, list the zeros of the polynomial and state the multiplicity of each zero.

7. $f(x) = x^{54}\left(x + \frac{4}{5}\right)$

9. $h(x) = 2x^{15}(x - \pi)^{14}[x - (\pi + 1)]^{13}$

In Exercises 11–22, find all the zeros of f in the complex number system; then write $f(x)$ as a product of linear factors.

11. $f(x) = x^2 - 2x + 5$

13. $f(x) = 3x^2 + 2x + 7$

15. $f(x) = x^3 - 27$ *Hint:* Factor first.

17. $f(x) = x^3 + 8$

19. $f(x) = x^4 - 1$

21. $f(x) = x^4 - 3x^2 - 10$

In Exercises 23–44, find a polynomial $f(x)$ with real coefficients that satisfies the given conditions. Some of the problems have many correct answers.

23. degree 3; only zeros are $1, 7, -4$

25. degree 6; only zeros are $1, 2, \pi$

27. degree 3; zeros $-3, 0, 4$; $f(5) = 80$

29. zeros include $2 + i$ and $2 - i$

31. zeros include 2 and $2 + i$

35. degree 2; zeros $1 + 2i$ and $1 - 2i$

37. degree 4; only zeros are 4, $3 + i$, and $3 - i$

39. degree 6; zeros 0 of multiplicity 3 and 3, $1 + i$, $1 - i$, each of multiplicity 1

41. degree 2; zeros include $1 + i$; $f(0) = 6$

43. degree 3; zeros include i and 1; $f(-1) = 8$

In Exercises 49–56, one zero of the polynomial is given; find all the zeros.

49. $x^3 - 2x^2 - 2x - 3$; zero 3

51. $x^4 + 3x^3 + 3x^2 + 3x + 2$; zero i

53. $x^4 - 2x^3 + 5x^2 - 8x + 4$; zero 1 of multiplicity 2

55. $x^4 - 4x^3 + 6x^2 - 4x + 5$; zero $2 - i$