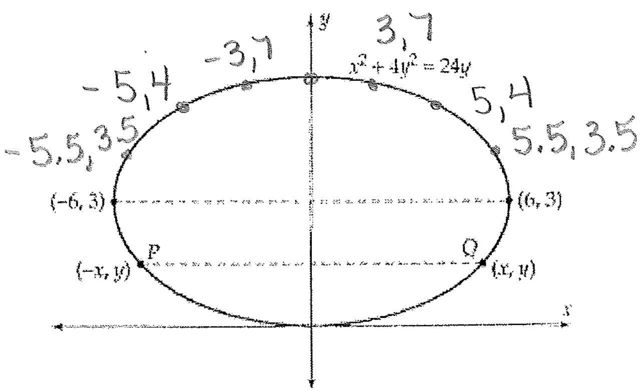
Name
Hour	Date

3.4.A Excursion: Symmetry

y-Axis Symmetry

A graph of a function or relation is **symmetric with respect to the y-axis** if the part of the graph on the right side of the y-axis is the mirror image of the part on the left side of the y-axis, as shown in Figure 3.4.A-1.



Each point P on the left side of the graph has a mirror image point Q on the right side of the graph, as indicated by the dashed lines. Note that

- their y-coordinates are the same
- their x-coordinates are opposites of each other
- the y-axis is the perpendicular bisector of $\overline{PQ}$

y-Axis Symmetry

A graph is symmetric with respect to the y-axis if whenever (x, y) is on the graph, then $(-x, y)$ is also on it.

In algebraic terms, this means that replacing x by $-x$ produces an equivalent equation.

To algebraically determine weather or not a function has symmetry to the y axis, simply **replace x for (-x)** and simplify it. If you get the same equation, then YES, it has symmetry to the y axis.

Example 1 y-Axis Symmetry

$x \to (-x)$

Verify that $y = x^4 - 5x^2 + 3$ is symmetric with respect to the y-axis.

$y = (-x)^4 - 5(-x)^2 + 3$

$y = x^4 - 5x^2 + 3$

yes

even function

x-Axis Symmetry

A graph of a relation is symmetric with respect to the x-axis if the part of the graph above the x-axis is the mirror image of the part below the x-axis, as shown in Figure 3.4.A-2.

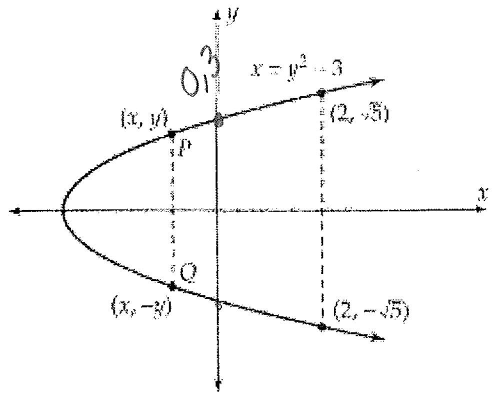


Figure 3.4.A-2

Each point P on the top of the graph has a mirror image point Q on the bottom of the graph, as indicated by the dashed lines. Note that

- their x-coordinates are the same
- their y-coordinates are opposites of each other
- the x-axis is the perpendicular bisector of $\overline{PQ}$

x-Axis Symmetry

A graph is symmetric with respect to the x-axis if whenever (x, y) is on the graph, then $(x, -y)$ is also on it.

In algebraic terms, this means that replacing y by $-y$ produces an equivalent equation.

Example 2 x-Axis Symmetry

Verify that $y^2 = 4x - 12$ is symmetric with respect to the x-axis.

$(-y)^2 = 4x - 12$

$y^2 = 4x - 12$

yes!

$y^3 = 4x - 12$

$(-y)^3 = 4x - 12$

$-y^3 = 4x - 12$

$y^3 = -4x + 12$

No!

To algebraically determine weather or not a function has symmetry to the x axis, simply **replace y for (-y) and simplify** it. If you get the same equation, then YES, it has symmetry to the x axis.

Origin Symmetry

A graph is **symmetric with respect to the origin** if a line through the origin and any point P on the graph also intersects the graph at a point Q such that the origin is the midpoint of $\overline{PQ}$, as shown in Figure 3.4.A-3.

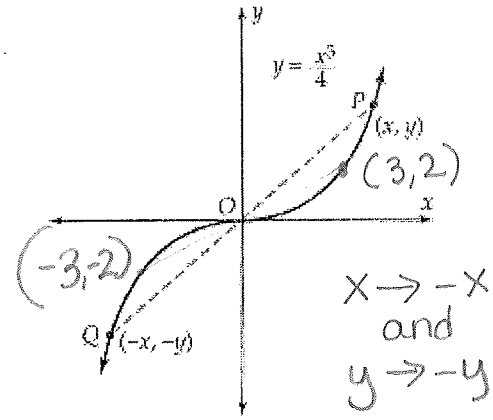


Figure 3.4.A-3

Using Figure 3.4.A-3, symmetry with respect to the origin can also be described in terms of coordinates and equations.

Origin Symmetry

A graph is symmetric with respect to the origin if whenever (x, y) is on the graph, then $(-x, -y)$ is also on it.

In algebraic terms, this means that replacing x by $-x$ and y by $-y$ produces an equivalent equation.

To algebraically determine weather or not a function has symmetry to the **origin**, simply replace **x or (-x)** and **y for (-y)** and simplify it. If you get the same equation, then YES, it has symmetry to the origin.

Also, take a look at the ordered pairs. For every point (x,y) there should be (-x, -y) on the graph. Then it has symmetry to the origin.

Example 3 Origin Symmetry

Verify that $y = \frac{x^3}{10} - x$ is symmetric with respect to the origin.

$$(-y) = \frac{(-x)^3}{10} - (-x)$$

$$-1\left(-y = \frac{-x^3}{10} + x\right)$$

$$y = \frac{x^3}{10} - x$$

yes origin
odd function

Symmetry Tests

Symmetry with respect to	Coordinate test	Algebraic test
y-Axis even funct.	If (x, y) is on the graph, then $(-x, y)$ is on the graph.	Replacing x by $-x$ produces an equivalent equation.
x-Axis	If (x, y) is on the graph, then $(x, -y)$ is on the graph.	Replacing y by $-y$ produces an equivalent equation.
Origin odd funct.	If (x, y) is on the graph, then $(-x, -y)$ is on the graph.	Replacing x by $-x$ and y by $-y$ produces an equivalent equation.

Even and Odd Functions

For relations that are *functions*, the algebraic description of symmetry can take a different form.

Even Functions

A function f whose graph is symmetric with respect to the y-axis is called an even function.

Even Functions

A function f is even if

$$f(-x) = f(x) \text{ for every value } x \text{ in the domain of } f.$$

The graph of an even function is symmetric with respect to the y-axis.

Odd Functions

A function f is odd if

$$f(-x) = -f(x) \text{ for every value } x \text{ in the domain of } f.$$

The graph of an odd function is symmetric with respect to the origin.

Exercises 3.4.A

In Exercises 1–6, graph the equation and state whether the graph has symmetry. If so, is it symmetric with respect to the x-axis, the y-axis, or the origin?

1. $y = x^2 + 2$ **2.** $x = (y - 3)^2$ **3.** $y = x^3 + 2$ **4.** $y = (x + 2)^3$

In Exercises 7–10, determine algebraically whether or not the graph of the given equation is symmetric with respect to the y-axis.

7. $x^2 + y^2 = 1$ **8.** $y = x^3 - x^2$ **9.** $4x^2 - 3y + y^2 = 7$ **10.** $x^4 + x^2 - x = y^3 + 1$

In Exercises 11–14, determine algebraically whether the graph of the given equation is symmetric with respect to the x-axis.

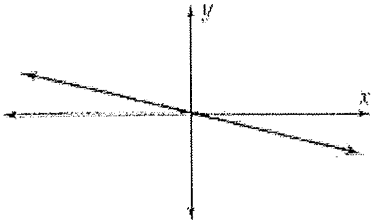
11. $x^2 - 6x + y^2 + 8 = 0$ **12.** $x^2 + 8x + y^2 = -15$ **13.** $x^2 - 2x + y^2 + 2y = 2$ **14.** $x^2 - x + y^2 - y = 0$

In Exercises 15–18, determine algebraically whether the graph of the given equation is symmetric with respect to the origin.

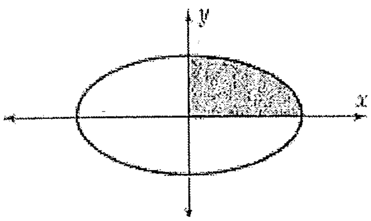
15. $4x^2 - 3y^2 + xy = 6$ **16.** $x^3 + y^3 = x$ **17.** $|x| + |y| = x^2 + y^2$ **18.** $3x^2 = 4y - 2x + 6$

In Exercises 19–24, determine whether the given graph is symmetric with respect to the y-axis, the x-axis, the origin, or any combination of the three.

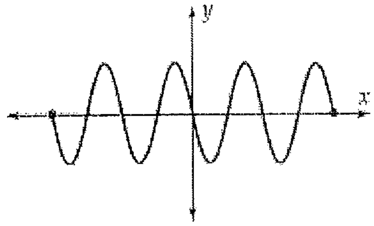
19.



20.



21.



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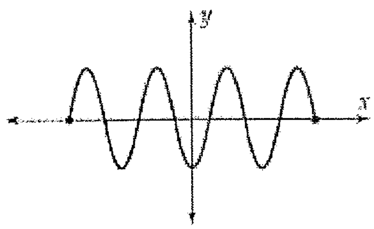

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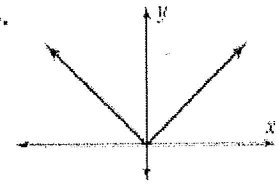

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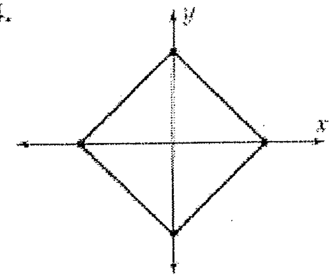


In Exercises 25–34, determine whether the given function is even, odd, or neither.

25. $f(x) = 4x$

26. $k(t) = -5t$

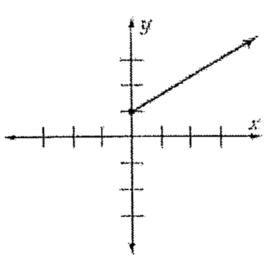
27. $f(x) = x^2 - |x|$

28. $h(u) = |3u|$

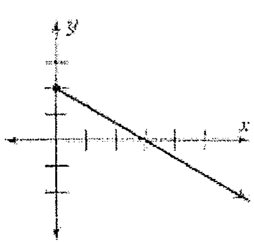
29. $k(t) = t^4 - 6t^2 + 5$

In Exercises 35–38, complete the graph of the given function, assuming that it satisfies the given symmetry condition.

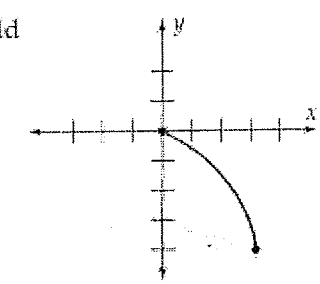
35. Even



36. Even



37. Odd



38. Odd